6.045J/18.400J: Automata, Computability and Complexity Prof. Nancy Lynch

Recitation 3: Solution Sketches

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Problem 1: **True or False?**

1. If L_1 and L_2 are regular, then $L_1 \cup L_2$ is regular. **True**
2. If L_1 and L_2 are non-regular, then $L_1 \cap L_2$ is non-regular.
   False. Consider $L_1 = \{0^n 1^n \mid n \geq 0\}$ and $L_2 = \{0^{n+1} 1^n \mid n \geq 0\}$.
3. If L_1 is regular and L_2 is non-regular, then $L_1 \cup L_2$ is non-regular.
   False. $L_1 = \Sigma^*$ and L_2 any non-regular language.
4. If L_1 is regular, L_2 is non-regular, and $L_1 \cap L_2$ is regular, than $L_1 \cup L_2$ is non-regular.
   True. Write $L_2 = \big\{(L_1 \cup L_2) - L_1\big\} \cup (L_1 \cap L_2)$.
5. The following language is regular: The set of strings in $\{0,1\}^*$ having the property that the number of 0's and the number of 1's differ by no more than 2.
   False.
6. The following language is regular: The set of strings in $\{0,1\}^*$ having the property that in every prefix, the number of 0's and the number of 1's differ by no more than 2.
   True. A simple 5-state DFA accepts this language.

Problem 2: **Regular Expressions.** Write regular expressions for the following languages. The alphabet is $\{0,1\}^*$.

1. $A_1 = \{w | w \text{ contains at least two 0's}\}$.
   Solution: $(0 \cup 1)^* 0 (0 \cup 1)^* 0 (0 \cup 1)^*$.
2. $A_2 = \{w | w \text{ contains an even number of 0's}\}$.
   Solution: $1^*(01^*01^*)^*$.

Problem 3: **Proving non-regularity: the Pumping Lemma.** Prove that the following languages are not regular.

1. $L_4 = \{0^i 1^j 2^k \mid i,j,k \geq 0 \text{ and if } i = 1 \text{ then } j = k\}$.
   Solution: Define $L_4' = \{1^j 2^k \mid j,k \geq 0\} \cup \{0^i 1^j 2^k \mid i > 1, j,k \geq 0\}$. It is easily seen that L_4' is regular. Now, observe that $L_4 - L_4' = \{01^j 2^j \mid j \geq 0\}$ is not regular.

Problem 4: **The size of the minimal DFA for a regular language L.** Consider the regular language $L = \{w \mid w \text{ contains at least three } 1's\}$. Prove that any DFA for this language has at least 4 states.

Solution: The crucial fact to use is that, if strings x and y lead from the start state to the same state q, then for every string z, $xz \in L$ if and only if $yz \in L$. More formally, $\delta^*(q_0, x) = \delta^*(q_0, y)$ implies $\forall z \in \Sigma^*, xz \in L$ if and only if $yz \in L$. (Think about it and convince yourself that this is true).

Now, note that strings $\epsilon, 1, 11, 111$ must lead to different states. For instance, suppose $\delta(q_0, \epsilon) = \delta(q_0, 1)$. Then, setting $z = 11$, we see that $11 \notin L$ whereas $111 \in L$. This is a contradiction, and therefore the strings ϵ and 1 lead to different states.