Problem Set 4

Due: October 27, 1995

Problem 1. We'll say two Scheme expressions M, N are *termination distinguishable* iff, in the initial Scheme environment, M halts and N does not, or vice versa. We'll say M and N are *functional termination congruent*, in symbols, $M \underset{\text{ftc}}{=} N$, iff $C[M]$ and $C[N]$ are not termination distinguishable for any functional Scheme context C. Explain why $\underset{\text{ftc}}{=}$ defines the *same* relation on expressions as $\underset{\text{func}}{=}$. (Remember, the definition of $\underset{\text{func}}{=}$ was like $\underset{\text{ftc}}{=}$ except M, N could be distinguished even when both halted, if they had different printable values.)

Problem 2. Let $\mathcal{S}_a$ be the set of Scheme expressions which do *not* evaluate to the symbol a.

2(a) Explain why $\mathcal{S}_a$ is a nontrivial, evaluation-invariant property of expressions (cf. Problem Set 3).

2(b) Explain how, given any Scheme expresssion, M, one can find a symbol, a, such that $M \in \mathcal{S}_a$.

2(c) Why doesn't 2(b) contradict the undecidability of nontrivial, evaluation-invariant properties?

2(d) Give another interesting example of a family of nontrivial evaluation-invariant properties, at least one of which can be found automatically for every Scheme expression.

Problem 3. With any equation $M = N$ we can associate two rewrite rules: $M \longrightarrow N$ and $N \longrightarrow M$. We can then regard any set of equations, $\mathcal{E}$, as a set of rewrite rules and use the notation $M \underset{\mathcal{E}}{\longrightarrow}^* N$ to mean that M can be rewritten to N by repeated use of the $\mathcal{E}$ rules. We use the notation $\underset{\mathcal{E}}{\vdash} M = N$ to mean there is a *formal equational proof* that $M = N$ in the system with axioms $\mathcal{E}$ and (reflexivity), and with inference rules (symmetry), (transitivity), (congruence).

3(a) Explain why $M \underset{\mathcal{E}}{\longrightarrow}^* N$ implies $\underset{\mathcal{E}}{\vdash} M = N$.

3(b) Explain how to transform a proof for $\underset{\mathcal{E}}{\vdash} M = N$ into a rewriting of M into N. Conclude that

$$M \underset{\mathcal{E}}{\longrightarrow}^* N \qquad \text{iff} \qquad \underset{\mathcal{E}}{\vdash} M = N.$$

3(c) Describe kinds of M and N for which the length of a proof that $M = N$ will be much shorter than a rewriting of M to N, where M, N are functional Scheme expressions and we use the equational Scheme axioms from class:

$M = N$	if $M \xrightarrow{\text{rew}} N$,
$((\lambda(x)E[x])M) = E[M]$	where E is an evaluation context and x is not free in E,
$M = N$	where N is obtained by instantiating a defined variable in M with another variable.

Problem 4.

4(a) Show that it is undecidable whether a Scheme expression halts in an *even number* of rewriting steps.

4(b) Does 4(a) follow from the undecidability of evaluation-invariant properties?

Problem 5. The intersection of two decidable sets is a decidable set. The intersection of two half-decidable (r.e.) sets is a half-decidable set. The complement of a half-decidable set in general is not half decidable. These facts are indicated by the "D," "H" and "N" entries in the table below. Complete the remaining table entries with D (decidable), H (half-decidable), or N (neither). Include brief justifications of each entry, including the given ones.

operation	decidable sets	half-decidable sets
intersection	D	H
union		
complement		N
cdr		