Diagnostic Quiz Solutions

Problem 1. ...qualifications...

Problem 2. Let succ be the successor function on integers:

$$\text{succ}(x) = x + 1$$

Describe the function (succ $\circ$ succ), that is, the composition of succ with itself.

The composition of succ with itself is the "add two" function, add2:

$$\text{add2}(x) = x + 2$$

because

$$\begin{aligned}(\text{succ} \circ \text{succ})(x) &= \text{succ}(\text{succ}(x)) \\ &= \text{succ}(x+1) \\ &= (x+1)+1 \\ &= x+2 \\ &= \text{add2}(x)\end{aligned}$$

Problem 3. How many strings of length 4 are there over the alphabet $\{a, b, c\}$?

With three possibilities in each of four positions, there are $3^4 = 81$ possible strings.

Problem 4. Define $\preceq$ to be the binary relation between sets such that $A \preceq B$ if and only if the cardinality of A is less than or equal to the cardinality of B.

(a) What is the definition of "uncountable set"? Now express the definition in terms of $\preceq$. Give an example of an uncountable set.

A set A is uncountable iff there is no one-to-one mapping from A into the integers. Thus, if $\mathbf{N}$ is the set of integers, then an uncountable set is any set A such that $A \not\preceq \mathbf{N}$. The real numbers, $\mathbb{R}$, are a well-known example of an uncountable set, as is the powerset of any infinite set.

(b) For each of the following properties, state whether the relation $\preceq$ has the property, and if not, give a simple counterexample.

1. reflexivity
2. symmetricity
3. transitivity

The relation $\preceq$ is reflexive ($A \preceq A$ for all sets A) and transitive (if $A \preceq B$ and $B \preceq C$ then $A \preceq C$), but not symmetric (there are sets A and B with $A \preceq B$ but $B \not\preceq A$, for example $\mathbf{N}$ and $\mathbb{R}$).

Problem 5. For any set, A, let $\mathcal{P}\text{ow}(A)$ be the powerset of A, namely, the set of all subsets of A. Exhibit the members of $\mathcal{P}\text{ow}(\mathcal{P}\text{ow}(\mathcal{P}\text{ow}(\emptyset)))$.

Given the empty set, $\emptyset$, with no members, the only subset is the (non-proper) subset $\emptyset$. Thus, if $\mathcal{P}\text{ow}(\emptyset)$ is the set of all subsets of $\emptyset$, then

$$\mathcal{P}\text{ow}(\emptyset) = \{\emptyset\}.$$

Moving along, the subsets of $\mathcal{P}\text{ow}(\emptyset)$ are $\emptyset$ and $\{\emptyset\}$, so

$$\mathcal{P}\text{ow}(\mathcal{P}\text{ow}(\emptyset)) = \{\emptyset, \{\emptyset\}\}.$$

Then, $\mathcal{P}\text{ow}(\mathcal{P}\text{ow}(\emptyset))$ has subsets $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, and $\{\emptyset, \{\emptyset\}\}$, so

$$\mathcal{P}\text{ow}(\mathcal{P}\text{ow}(\mathcal{P}\text{ow}(\emptyset))) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$$

Problem 6. Two Boolean formulas $F_1(x_1, \ldots, x_n)$ and $F_2(x_1, \ldots, x_n)$ are *equivalent* iff they yield the same truth value for all truth assignments to the variables $x_1, \ldots, x_n$.

(a) ... Describe a Boolean binary operation which is not commutative.

The "implies" operation, often written $\Rightarrow$, is non-commutative: $\mathbf{false} \Rightarrow \mathbf{true}$, but $\mathbf{true} \not\Rightarrow \mathbf{false}$, so $x_1 \Rightarrow x_2$ and $x_2 \Rightarrow x_1$ are not equivalent.

(b) Describe an infinite set of equivalent Boolean formulas.

Define the formulas F_i as $F_0 \stackrel{\text{def}}{=} x_0$ and $F_{i+1} \stackrel{\text{def}}{=} (F_i \;\&\; x_0)$ for all $i \geq 0$. These formulas are all equivalent, since for any truth assignment they all have the same truth value as x_0.

(c) Explain why "equivalent" is actually an equivalence relation on formulas.

An equivalence relation is a binary relation on a set A, that is, $R \subseteq A \times A$, which is reflexive ($a\ R\ a$ for all $a \in A$), transitive (if $a\ R\ a'$ and $a'\ R\ a''$ then $a\ R\ a''$), and symmetric ($a\ R\ a'$ iff $a'\ R\ a$).

Take any three formulas F_0, F_1 and F_2. Given any truth assignment to $x_1, \ldots, x_n$:

- F_0 yields the same truth value as itself, so "equivalent" is reflexive.
- If F_0 yields the same truth value as F_1, then F_1 yields the same truth value as F_0, so "equivalent" is symmetric.
- If F_0 yields the same truth value as F_1, and F_1 yields the same truth value as F_2, then clearly F_0 yields the same truth value as F_2, so "equivalent" is transitive.

Thus, "equivalent" is an equivalence relation on formulas.

(d) Explain why there are only a finite number of equivalence classes of formulas with (at most) variables $x_1, \ldots, x_n$.

(e) How many?

Formulas are equivalent iff their truth tables agree, so there are as many equivalence classes as there are truth tables. Given n variables, there are 2^{2^n} possible truth tables. To see this, think of a truth table as having a row for each possible truth assignment. A truth assignment consists of a **true** or **false** value for each variable, so there are (size of $\{\mathbf{true}, \mathbf{false}\})^n = 2^n$ possible truth assignments. Then, a truth table consists of an assignment of **true** or **false** to each truth assignment, so with 2^n truth assignments there are 2^{2^n} possible truth tables, giving us 2^{2^n} equivalence classes of formulas.

Problem 7. A binary relation $\leq$ is *anti-symmetric* if $x = y$ whenever $x \leq y$ and $y \leq x$. A *partial order* of a set D is a binary relation $\leq$ that is reflexive, transitive, and anti-symmetric on D. A partial order is *total* if for every $d_1, d_2 \in D$, either $d_1 \leq d_2$ or $d_2 \leq d_1$. If $X \subseteq D$ and $y \leq x$ for every $x \in X$, then y is a *lower bound* of X. Similarly, if $x \leq y$ for every $x \in X$, then y is an *upper bound* of X. If y is a lower bound of X, and $z \leq y$ for every lower bound z of X, then y is the *greatest lower bound* ("glb" or "meet") of X. If y is an upper bound of X, and $y \leq z$ for every upper bound z of X, then y is the *least upper bound* ("lub") of X.

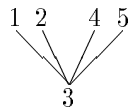
Let A be the set $\{1, 2, 3, 4, 5\}$.

(a) Give a total order of A. That is, describe a binary relation $\leq$ on A that is a total order.

The usual numerical "less than or equal to" relation is a total order.

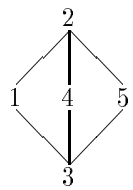
(b) Give a partial order of A such that: (1) every pair of elements has a glb; and (2) there is no lub of $\{4, 5\}$.

The ordering in which 3 is less than or equal to any other number, but all other numbers are incomparable, is such an order. In pictures, we can draw the order as follows:



Here we are using a common graphical shorthand for partial orders, in which an element is "less than" another element if we can trace an upwards path from the first element to the second. Since there is an upwards path from 3 to every other number, we have $3 \leq x$ for all $x \in A$. (There is an upwards path from 3 to 3, namely the upwards path of length 0). However, $1 \not\leq 4$ since there is no upwards path from 1 to 4, and $4 \not\leq 1$ since there is no upwards path from 4 to 1; that is, 1 and 4 are incomparable.

From the diagram, it is clear that every pair of distinct numbers has a unique lower bound in the ordering, namely 3. Since there is only one lower bound possible, it is clearly the greatest lower bound. However, 4 and 5 have no upper bound at all, let alone a least upper bound.

(c) Give a partial order of A such that every pair of elements has both an lub and a glb, but the order is not total.

A slight variant, in which we leave 3 as the least element but make 2 the greatest, gives us an ordering in which every pair of elements has both a least upper bound and a greatest lower bound, but the ordering is still not total.