State Machines

State machines

step by step processes
(may step in response to input —not today)

The state graph of a 99-bounded counter:

- States: \(\{0, 1, \ldots, 99, \text{overflow}\}\)
- Transitions: \(i \rightarrow i+1\) for \(0 \leq i < 99\)
- \(99 \rightarrow \text{overflow}\)
- \(\text{overflow} \rightarrow \text{overflow}\)

Die Hard

Picture source: http://movieweb.com/movie/diehard3/
Simon says: On the fountain, there should be 2 jugs, do you see them? A 5-gallon and a 3-gallon. Fill one of the jugs with exactly 4 gallons of water and place it on the scale and the timer will stop. You must be precise; one ounce more or less will result in detonation. If you're still alive in 5 minutes, we'll speak.

Supplies:
- 3 Gallon Jug
- 5 Gallon Jug

Transferring water:
- 3 Gallon Jug
- 5 Gallon Jug
Die hard state machine

State:
amount of water in jugs: \((b,l)\)
\[0 \leq b \leq 5, \quad 0 \leq l \leq 3\]
Start State: \((0,0)\)

State machines

Die Hard Transitions:
1. Fill little jug: \((b, l) \rightarrow (b, 3)\) for \(l < 3\)
2. Fill big jug: \((b, l) \rightarrow (5, l)\) for \(b < 5\)
3. Empty little jug: \((b, l) \rightarrow (b, 0)\) for \(l > 0\)
4. Empty big jug: \((b, l) \rightarrow (0, l)\) for \(b > 0\)
5. Pour big jug into little jug
   (i) If no overflow, then \((b, l) \rightarrow (0, b+l)\)
       \[b+l \leq 3\]
   (ii) otherwise \((b, l) \rightarrow (b-(3-l),3)\)
6. Pour little jug into big jug.
   Likewise

Simon's challenge:
Disarm the bomb by putting precisely 4 gallons of water on the scale, or it will blow up.
(You can figure out how)
Work it out now!

How to do it

Start with empty jugs: (0,0)
Fill the big jug: (5,0)

Pour from big to little: (2,3)

Empty the little: (2,0)
How to do it

Pour from big to little: (0,2)

3 Gallon Jug 5 Gallon Jug

Fill the big jug: (5,2)

3 Gallon Jug 5 Gallon Jug

How to do it

Pour from big to little: (4,3)

3 Gallon Jug 5 Gallon Jug

Done!

Die Hard once and for all

What if have a 9 gallon jug instead?

3 Gallon Jug 5 Gallon Jug 9 Gallon Jug

Can you do it? Can you prove it?
Preserved Invariants

Die hard once and for all preserved invariant:

\[ P(\text{state}) ::= \text{“3 divides the number of gallons in each jug.”} \]

\[ P((b,l)) ::= (3 \mid b \text{ AND } 3 \mid l) \]

Die Hard Once & For All

Corollary: No state \((4,x)\) is reachable, so Bruce Dies!

Floyd’s Invariant Principle

(Induction for state machines)

Preserved Invariant, \(P(\text{state})\):

If \(P(q)\) and \(q \rightarrow r\), then \(P(r)\)

Conclusion: if \(P(\text{start})\), then \(P(r)\)

for all reachable states \(r\), including final state (if any)
The Diagonal Robot

the robot is on a grid

\[ \begin{array}{|c|c|c|c|}
\hline
& 0 & 1 & 2 \\
\hline
0 & & & \\
\hline
1 & & & \\
\hline
2 & & & \\
\hline
\end{array} \]

\[ \text{GOAL} \]

it can move diagonally

\[ \begin{array}{|c|c|c|c|}
\hline
& 0 & 1 & 2 \\
\hline
0 & & & \\
\hline
1 & & & \\
\hline
2 & & & \\
\hline
\end{array} \]

\[ \times \times \times \]

Robot Preserved Invariant

\[ P((x, y)) := x + y \text{ is even} \]

move adds \( \pm 1 \) to both \( x \) & \( y \), preserving parity of \( x+y \).

Also, \( P((0, 0)) \) is true.
 Robot Preserved Invariant

So in all positions \((x,y)\) reachable from \((0,0)\),
\(x + y\) stays even
But \(1 + 0 = 1\) is odd, so
\((1,0)\) is not reachable

The Fifteen Puzzle
Explained!

--by similar reasoning
details in problem 2

Fast Exponentiation
\(a^b\) using registers \(X, Y, Z, R\)

\[
\begin{align*}
X &:= a; \quad Y := 1; \quad Z := b; \\
\text{REPEAT:} & \\
\text{if Z=0, then return Y} & \\
R := \text{remdr}(Z,2); \quad Z := \text{quotnt}(Z,2) & \\
\text{if R=1, then Y := X \cdot Y} & \\
X &:= X^2
\end{align*}
\]
Fast Exponentiation

Preserved Invariant: $YX^Z = a^b$

$(X,Y,Z) \rightarrow [Z>0 \text{ is odd}]
(X^2, X \cdot Y, (Z-1)/2)$

$(X \cdot Y)(X^2)^{(Z-1)/2} = (X \cdot Y)X^{Z-1}$

$= YX^Z = a^b$

Partial Correctness

preserved invariant: $YX^Z = a^b$

at start $1 \cdot a^b = a^b$

at end $Z=0$, so return

$Y=YX^0 = a^b$

Fast Termination

at each transition

$Z := \text{quotient}(Z,2)$

$Z = b$ at start, so $Z = 0$

in $\leq \log_2(b)$ transitions

Robert W Floyd (1934–2001)
