Recursive Functions

To define a function, $f$, on a recursively defined set $R$, define

- $f(b)$ explicitly for each base case $b \in R$
- $f(c(x))$ for each constructor, $c$, in terms of $x$ and $f(x)$

Recursive function on $M$

Def. tree-depth($s$) for $s \in M$

$td(\lambda) ::= 0$

$td([s]t) ::= 1 + \max\{td(s), td(t)\}$

$k^n$ — recursive function on $\mathbb{N}$

$\text{expt}(k, 0) ::= 1$

$\text{expt}(k, n+1) ::= k \cdot \text{expt}(k, n)$

--uses recursive def of $\mathbb{N}$:

- $0 \in \mathbb{N}$
- if $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$
Recursive Functions

summary:
f: Data → Values
f(b) def’d directly for base b
f(cnstr(x)) def’d using f(x), x

Length versus Depth

Lemma: |r| + 2 ≤ 2^{td(r)+1}
for all r ∈ M
Proof by Structural Induction
Base case: [r = λ]
|λ|+2 = 0+2 = 2 = 2^{0+1} = 2^{td(λ)+1}
OK!

Size versus Depth
Constructor case: [r = [s]t]
by ind. hypothesis:

|s| + 2 ≤ 2^{td(s)+1}
|t| + 2 ≤ 2^{td(t)+1}

Size versus Depth

|r| + 2 = |[s]| + 2\text{ def. of }r
= (|s|+|t|+2)+2\text{ def. of length}
= (|s|+2)+(|t|+2)
≤ 2^{td(s)+1} + 2^{td(t)+1}\text{ induction hyp.}
= 2\cdot2\max(td(s),td(t))+1 + 2\max(td(s),td(t))+1
= 2\cdot2\max(td(s),td(t))+1 \leq 2\cdot2^{td(r)}\text{ def. of }d(r)
= 2^{td(r)+1}\text{ QED!}
positive powers of two

$2 \in \text{PP}_2$

if $x, y \in \text{PP}_2$, then $x \cdot y \in \text{PP}_2$

$2, 2 \cdot 2, 4 \cdot 2, 4 \cdot 4, 4 \cdot 8, \ldots$

$2, 4, 8, 16, 32 \ldots \in \text{PP}_2$

log$_2$ of PP2

log$_2$(2) ::= 1

log$_2$(x \cdot y) ::= \log$_2$(x) + \log$_2$(y)

for $x, y \in \text{PP}_2$

log$_2$(4) = log$_2$(2 \cdot 2) = 1 + 1 = 2

log$_2$(8) = log$_2$(2 \cdot 4) = log$_2$(2) + log$_2$(4)

= 1 + 2 = 3

log$_2$ function on PP2

log$_2$(4) = log$_2$(2 \cdot 2) = 1 + 1 = 2

log$_2$(8) = log$_2$(2 \cdot 4) = log$_2$(2) + log$_2$(4)

= 1 + 2 = 3

log$_2$ function on PP2

log$_2$(16) = log$_2$(8 \cdot 2) = 3 + 1 = 4

log$_2$(16) = log$_2$(2 \cdot 8)

= 2 + log$_2$(8) = 2 + 3

= 5

log$_2$(16) = log$_2$(8 \cdot 2) = 8 + log$_2$(2)

= 3 + 1 = 4

log$_2$(16) = log$_2$(2 \cdot 8)

= 2 + log$_2$(8) = 2 + 3

= 5
ambiguous constructors

The Problem: more than one way to construct elements of PP2 from
\( \text{cnstrct}(x,y) = x \cdot y \)

16 = \text{cnstrct}(8,2) but also
16 = \text{cnstrct}(2,8)

ambiguous recursive defs

problem to watch out for:
recursive function on datum, \( e \),
is defined according to what constructor created \( e \).
If 2 or more ways to construct \( e \),
then which definition to use?