


**Mathematics for Computer Science**  
 MIT 6.042J/18.062J

# Asymptotic Notation



 Albert R Meyer, November 6, 2009 lec 9F.1



**Closed form for  $n!$**

$$n! ::= 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n = \prod_{i=1}^n i$$

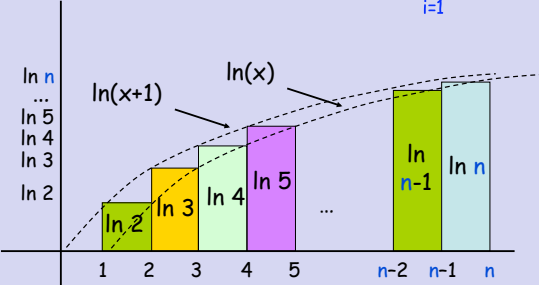
Turn product into a **sum** taking logs:


$$\begin{aligned} \ln(n!) &= \ln(1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n) = \\ &= \ln 1 + \ln 2 + \cdots + \ln(n-1) + \ln(n) \\ &= \sum_{i=1}^n \ln(i) \end{aligned}$$



 Albert R Meyer, November 6, 2009 lec 9F.2


**Closed form for  $n!$**

Integral Method to bound  $\sum_{i=1}^n \ln(i)$





 Albert R Meyer, November 6, 2009 lec 9F.3



**Closed form for  $n!$**

$$\begin{aligned} n \ln\left(\frac{n}{e}\right) + 1 &\leq \sum_{i=1}^n \ln(i) \\ &\leq (n+1) \ln\left(\frac{n+1}{e}\right) + 0.6 \end{aligned}$$

reminder:

$$\int \ln x \, dx = x \ln\left(\frac{x}{e}\right)$$



 Albert R Meyer, November 6, 2009 lec 9F.5



**Closed form for  $n!$**

$$\sum_{i=1}^n \ln(i) \approx \left(n + \frac{1}{2}\right) \ln\left(\frac{n}{e}\right)$$

exponentiating:


$$n! \approx \sqrt{n/e} \left(\frac{n}{e}\right)^n$$



 Albert R Meyer, November 6, 2009 lec 9F.6



**Stirling's Formula**


A precise approximation:


$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$



 Albert R Meyer, November 6, 2009 lec 9F.7



**Little Oh:  $o(\cdot)$**   
 Asymptotically smaller :  
 Def:  $f(n) = o(g(n))$   
 iff  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$


 Albert R Meyer, November 6, 2009 lec 9F.13



**Little Oh:  $o(\cdot)$**   
 $n^2 = o(n^3)$   
 because  
 $\lim_{n \rightarrow \infty} \frac{n^2}{n^3} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$


 Albert R Meyer, November 6, 2009 lec 9F.14



**Big Oh:  $O(\cdot)$**   
 Asymptotic Order of Growth:  
 $f(n) = O(g(n))$   
~~limsup~~  $\limsup_{n \rightarrow \infty} \left( \frac{f(n)}{g(n)} \right) < \infty$   
 a technicality -- ignore now


 Albert R Meyer, November 6, 2009 lec 9F.15


**Big Oh:  $O(\cdot)$**   
 $3n^2 = O(n^2)$   
 because  
 $\lim_{n \rightarrow \infty} \frac{3n^2}{n^2} = 3 < \infty$


 Albert R Meyer, November 6, 2009 lec 9F.16


**Theta:  $\Theta(\cdot)$**   
 Same Order of Growth:  
 $f(n) = \Theta(g(n))$   
 Def:  $f(n) = O(g(n))$   
 and  
 $g(n) = O(f(n))$

 Albert R Meyer, November 6, 2009 lec 9F.17


**Asymptotics: Intuitive Summary**

- $f \sim g$ :  $f$  &  $g$  nearly equal
- $f = o(g)$ :  $f$  much less than  $g$
- $f = O(g)$ :  $f$  roughly  $\leq g$
- $f = \Theta(g)$ :  $f$  &  $g$  roughly equal

 Albert R Meyer, November 6, 2009 lec 9F.18

**The Oh's**

lemma:  
 If  $f = o(g)$  or  $f \sim g$ , then  $f = O(g)$   
 $\lim = 0$  or  $\lim = 1$  IMPLIES  $\lim < \infty$

Albert R Meyer, November 6, 2009 lec 9F.19

**The Oh's**

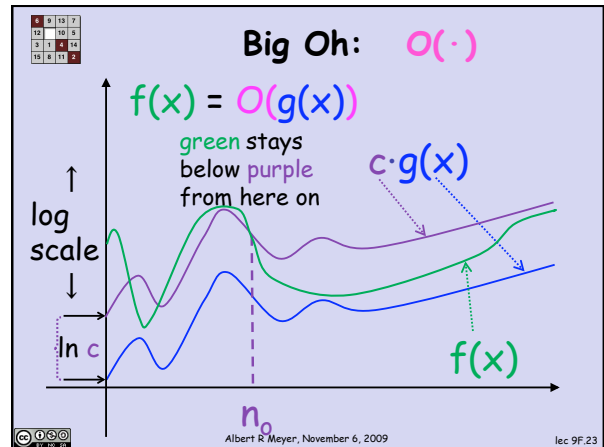
If  $f = o(g)$ , then  $g \neq O(f)$   
 $\lim \frac{f}{g} = 0$  IMPLIES  $\lim \frac{g}{f} = \infty$

Albert R Meyer, November 6, 2009 lec 9F.21

**Big Oh:  $O(\cdot)$**

Equivalent definition:  
 $f(n) = O(g(n))$   
 $\exists c, n_0 \forall n \geq n_0.$   
 $f(n) \leq c \cdot g(n)$

Albert R Meyer, November 6, 2009 lec 9F.22



**Little Oh:  $o(\cdot)$**

Lemma:  $x^a = o(x^b)$  for  $a < b$

Proof:  $\frac{x^a}{x^b} = \frac{1}{x^{b-a}}$  and  $b - a > 0$   
 so as  $x \rightarrow \infty$ ,  $\frac{1}{x^{b-a}} \rightarrow 0$

Albert R Meyer, November 6, 2009 lec 9F.24

**Little Oh:  $o(\cdot)$**

Lemma:  $\ln x = o(x^\epsilon)$   
 for  $\epsilon > 0$ .

Albert R Meyer, November 6, 2009 lec 9F.25

6	9	13	7
12	10	5	
3	1	14	11
15	8	11	2

Little Oh:  $o(\cdot)$

Lemma:

$$x^n = o(a^x)$$

for  $a > 1$ .



Albert R Meyer, November 6, 2009

lec 9F.29

6	9	13	7
12	10	5	
3	1	14	11
15	8	11	2

Little Oh:  $o(\cdot)$

proofs:

L'Hopital's Rule,  
McLaurin Series  
(see a Calculus text)



Albert R Meyer, November 6, 2009

lec 9F.30

6	9	13	7
12	10	5	
3	1	14	11
15	8	11	2

Big Oh **Mistakes**

" $\cdot = O(\cdot)$ " defines a relation

Don't write  $O(g) = f$ .

Otherwise:  $x = O(x)$ , so  $O(x) = x$ .

But  $2x = O(x)$ , so

$$2x = O(x) = x,$$

therefore  $2x = x$ .

Nonsense!



Albert R Meyer, November 6, 2009

lec 9F.31

6	9	13	7
12	10	5	
3	1	14	11
15	8	11	2

Team Problems

# Problems

# 1-4



Albert R Meyer, November 6, 2009

lec 9F.35