*Mathematics for Computer Science*
MIT 6.042J/18.062J

# Asymptotic Notation



## Closed form for n!

$$n! ::= 1\cdot 2\cdot 3\cdots(n-1)\cdot n = \prod_{i=1}^{n} i$$

Turn product into a sum taking logs:

$$\ln(n!) = \ln(1\cdot 2\cdot 3\cdots(n-1)\cdot n) =$$

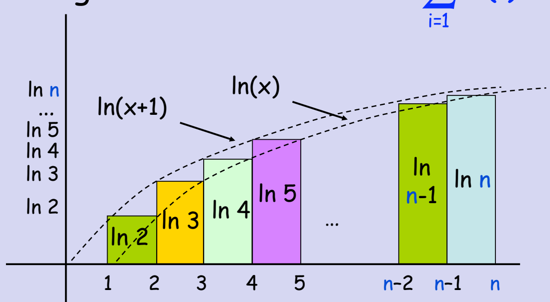
$$\ln 1 + \ln 2 + \cdots + \ln(n-1) + \ln(n)$$

$$= \sum_{i=1}^{n} \ln(i)$$



## Closed form for n!

Integral Method to bound $\sum_{i=1}^{n} \ln(i)$






## Closed form for n!

$$n\ln\left(\frac{n}{e}\right)+1 \leq \sum_{i=1}^{n}\ln(i)$$

$$\leq (n+1)\ln\left(\frac{n+1}{e}\right)+0.6$$

*reminder:*

$$\int \ln x\,dx = x\ln\left(\frac{x}{e}\right)$$



## Closed form for n!

$$\sum_{i=1}^{n}\ln(i) \approx (n+\frac{1}{2})\ln\left(\frac{n}{e}\right)$$

exponentiating:

$$n! \approx \sqrt{n/e}\left(\frac{n}{e}\right)^{n}$$



## Stirling's Formula

A precise approximation:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}$$

## Little Oh: $o(\cdot)$

Asymptotically smaller :

Def: $f(n) = o(g(n))$

iff $$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$



## Little Oh: $o(\cdot)$

$$n^2 = o(n^3)$$

because

$$\lim_{n\to\infty} \frac{n^2}{n^3} = \lim_{n\to\infty} \frac{1}{n} = 0$$



## Big Oh: $O(\cdot)$

Asymptotic Order of Growth:

$$f(n) = O(g(n))$$

$$\lim ~~\text{sup}~~_{n\to\infty} \left(\frac{f(n)}{g(n)}\right) < \infty$$

a technicality -- ignore now



## Big Oh: $O(\cdot)$

$$3n^2 = O(n^2)$$

because

$$\lim_{n\to\infty} \frac{3n^2}{n^2} = 3 < \infty$$



## Theta: $\Theta(\cdot)$

Same Order of Growth:

$$f(n) = \Theta(g(n))$$

Def: $f(n)=O(g(n))$

and

$g(n)=O(f(n))$



## Asymptotics: Intuitive Summary

| | |
|---|---|
| $f \sim g$: | f & g nearly equal |
| $f = o(g)$: | f much less than g |
| $f = O(g)$: | f roughly $\leq$ g |
| $f = \Theta(g)$: | f & g roughly equal |

## The Oh's

lemma:

If $f = o(g)$ or $f \sim g$, then $f = O(g)$

$\lim = 0$ or $\lim = 1$ IMPLIES $\lim < \infty$



## The Oh's

If $f = o(g)$, then $g \neq O(f)$

$\lim \frac{f}{g} = 0$ IMPLIES $\lim \frac{g}{f} = \infty$



## Big Oh: $O(\cdot)$

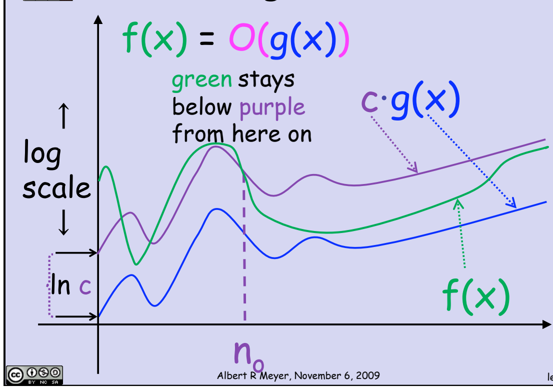
Equivalent definition:

$$f(n) = O(g(n))$$

$$\exists c, n_0\ \forall n \geq n_0.\ f(n) \leq c \cdot g(n)$$



## Big Oh: $O(\cdot)$






## Little Oh: $o(\cdot)$

Lemma: $x^a = o(x^b)$ for $a < b$

Proof: $\frac{x^a}{x^b} = \frac{1}{x^{b-a}}$ and $b - a > 0$

so as $x \to \infty$, $\frac{1}{x^{b-a}} \to 0$



## Little Oh: $o(\cdot)$

Lemma:

$$\ln x = o(x^{\varepsilon})$$

for $\varepsilon > 0$.

## Little Oh: $o(\cdot)$

Lemma:

$$x^n = o(a^x)$$

for $a > 1$.



## Little Oh: $o(\cdot)$

proofs:
L'Hopital's Rule,
McLaurin Series
(see a Calculus text)



## Big Oh Mistakes

"$\cdot = O(\cdot)$" defines a relation

Don't write $O(g) = f$.

Otherwise: $x = O(x)$, so $O(x) = x$.

But $2x = O(x)$, so

$2x = O(x) = x$,

therefore $2x = x$.

Nonsense!



## Team Problems

# Problems

# 1–4