Problem Set 8

Due: November 6

Reading: Notes Ch. 13.4–13.7; Ch. 14.1–14.4

Problem 1.
Find an integer $k > 1$ such that n and n^k agree in their last three digits whenever n is divisible by neither 2 nor 5. *Hint:* Euler's theorem.

Problem 2.
Find the remainder of $26^{1818181}$ divided by 297. *Hint:* $1818181 = (180 \cdot 10101) + 1$; Euler's theorem

Problem 3.
Suppose m, n are relatively prime. In the problem you will prove the key property of Euler's function that $\phi(mn) = \phi(m)\phi(n)$.

(a) Prove that for any a, b, there is an x such that

$$x \equiv a \pmod{m}, \tag{1}$$
$$x \equiv b \pmod{n}. \tag{2}$$

Hint: Congruence (1) holds iff

$$x = jm + a. \tag{3}$$

for some j. So there is such an x only if

$$jm + a \equiv b \pmod{n}. \tag{4}$$

Solve (4) for j.

(b) Prove that there is an x satisfying the congruences (1) and (2) such that $0 \leq x < mn$.

(c) Prove that the x satisfying part (b) is unique.

(d) For an integer k, let k^* be the integers between 1 and $k - 1$ that are relatively prime to k. Conclude from part (c) that the function

$$f : (mn)^* \to m^* \times n^*$$

defined by

$$f(x) ::= (\text{rem}(x, m), \text{rem}(x, n))$$

is a bijection.

(e) Conclude from the preceding parts of this problem that

$$\phi(mn) = \phi(m)\phi(n).$$

Problem 4.
There is a bug on the edge of a 1-meter rug. The bug wants to cross to the other side of the rug. It crawls at 1 cm per second. However, at the end of each second, a malicious first-grader named Mildred Anderson *stretches* the rug by 1 meter. Assume that her action is instantaneous and the rug stretches uniformly. Thus, here's what happens in the first few seconds:

- The bug walks 1 cm in the first second, so 99 cm remain ahead.
- Mildred stretches the rug by 1 meter, which doubles its length. So now there are 2 cm behind the bug and 198 cm ahead.
- The bug walks another 1 cm in the next second, leaving 3 cm behind and 197 cm ahead.
- Then Mildred strikes, stretching the rug from 2 meters to 3 meters. So there are now $3 \cdot (3/2) = 4.5$ cm behind the bug and $197 \cdot (3/2) = 295.5$ cm ahead.
- The bug walks another 1 cm in the third second, and so on.

Your job is to determine this poor bug's fate.

(a) During second i, what *fraction* of the rug does the bug cross?

(b) Over the first n seconds, what fraction of the rug does the bug cross altogether? Express your answer in terms of the Harmonic number H_n.

(c) The known universe is thought to be about $3 \cdot 10^{10}$ light years in diameter. How many universe diameters must the bug travel to get to the end of the rug?

Problem 5.
Let f, g be nonnegative real-valued functions such that $\lim_{x \to \infty} f(x) = \infty$ and $f \sim g$.

(a) Give an example of f, g such that $\text{NOT}(2^f \sim 2^g)$.

(b) Prove that $\log f \sim \log g$.

(c) Use Stirling's formula to prove that in fact

$$\log(n!) \sim n \log n$$

Student's Solutions to Problem Set 8

Your name:

Due date: November 6

Submission date:

Circle your TA/LA: Jodyann Justin Megumi Rajeev Richard Steven Tom Eli

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.
2. I collaborated on this assignment with:

   got help from:[1]

   and referred to:[2]

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	
2	
3	
4	
5	
Total	

[1]People other than course staff.
[2]Give citations to texts and material other than the Fall '09 course materials.