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Notes for Recitation 8

1 The Grow Algorithm

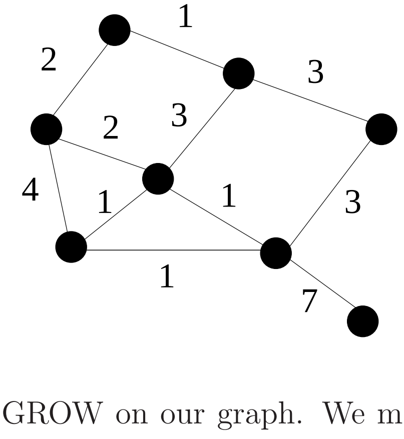
Yesterday in lecture, we saw the following algorithm for constructing a minimum-weight spanning tree (MST) from an edge-weighted N-vertex graph G.

ALG-GROW:

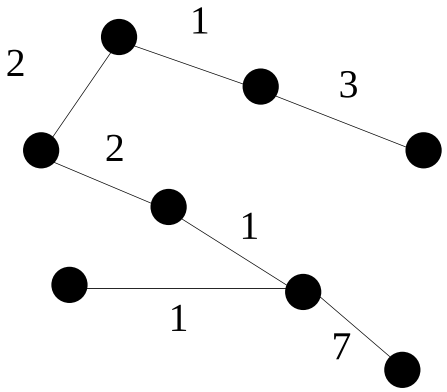
1. Label the edges of the graph $e_1, e_2, \ldots, e_t$ so that $wt(e_1) \leq wt(e_2) \ldots \leq wt(e_t)$.
2. Let S be the empty set.
3. For $i = 1 \ldots t$, if $S \cup \{e_i\}$ does not contain a cycle, then extend S with the edge e_i.
4. Output S.

In summary, ALG-GROW selects edges one at a time, always choosing the minimum weight edge that does not create a cycle with previously selected edges. Notice that as edges are added S may not be connected. When the algorithm terminates, S contains $N-1$ edges. If it is connected, then it is a spanning tree.

Consider, for example, the following edge-weighted graph.



Now suppose we run ALG-GROW on our graph. We may choose the weight 1 edge on the bottom of the triangle of weight 1 edges in our graph. In the next step, we may choose the weight 1 edge on the top of the graph. Note that this edge still has minimum weight, and does not cause us to form a cycle, so ALG-GROW can choose it. We will then choose one of the remaining weight 1 edges. Note that neither causes us to form a cycle. Continuing the algorithm, we may end up with the same spanning tree shown below.



In this recitation, we will analyze ALG-GROW.

2 Analysis of ALG-GROW

In this problem you may assume the following lemma from the problem set:

Lemma 1. *Suppose that $T = (V, E)$ is a simple, connected graph. Then T is a tree iff $|E| = |V| - 1$.*

In this exercise you will prove the following theorem.

Theorem. *For any connected, weighted graph G, ALG-GROW produces an MST of G.*

(a) Prove the following lemma.

Lemma 2. *Let $T = (V, E)$ be a tree and let e be an edge not in E. Then, $G = (V, E \cup \{e\})$ contains a cycle.*

(Hint: Suppose G does *not* contain a cycle. Is G a tree?)

Solution. *Proof.* (by contradiction) Suppose G does not contain a cycle. By the definition of a tree, T is connected. Notice that T is a subgraph of G. Because any two nodes in G are connected by a path in T, G is a connected graph. So G is connected and acyclic and therefore a tree by definition. Both G and T are trees and have the same number of nodes. Therefore, they have the same number of edges (by Lemma 1). This is a contradiction because G has one more edge than T. □

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(b) Prove the following lemma.

Lemma 3. *Let $T = (V, E)$ be a spanning tree of G and let e be an edge not in E. Then there exists an edge $e' \neq e$ in E such that $T^* = (V, E - \{e'\} \cup \{e\})$ is a spanning tree of G.*

(Hint: Adding e to E introduces a cycle in $(V, E \cup \{e\})$.)

Solution. *Proof.* By Lemma 2, we know that the set of edges $E \cup \{e\}$ contains a cycle. If this cycle does not contain the edge e, then this cycle is a subset of E. Since E is the set of edges of a tree, this cannot occur. So, this cycle contains e. If e' is another edge distinct from e in this cycle, then the graph T^* that results after removing e' from $E \cup \{e\}$ is still connected. The number of edges in T^* is equal to the number of edges in T, which is equal to $|V| - 1$ by Lemma 1. Since T^* is connected, T^* is a tree by Lemma 1. Since T^* is a subgraph of G with vertices V, it spans G. □

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(c) Prove the following lemma.

Lemma 4. *Let $T = (V, E)$ be a spanning tree of G, let e be an edge not in E and let $S \subseteq E$ such that $S \cup \{e\}$ does not contain a cycle. Then there exists an edge $e' \neq e$ in $E - S$ such that $T^* = (V, E - \{e'\} \cup \{e\})$ is a spanning tree of G.*

(Hint: Modify your proof to part (b). Of all possible edges $e' \neq e$ that can be removed to construct T^*, at least one is not in S.)

Solution. *Proof.* We need to change the proof in part (b) slightly. The proof of part (b) holds for any edge $e' \neq e$ in the cycle. We need to show that we can select an edge $e' \neq e$ that is in the cycle but not in S. We will prove this by contradiction. Suppose that all the edges not equal to e that are in the cycle are in S. Then, $S \cup e$ is a cycle. This contradicts the assumption of the lemma. □

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(d) Prove the following lemma.

Lemma 5. *Define S_m to be the set consisting of the first m edges selected by ALG-GROW from a connected graph G. Let $P(m)$ be the predicate that if $m \leq |V|$ then $S_m \subseteq E$ for some MST $T = (V, E)$ of G. Then $\forall m \,.\, P(m)$.*

(Hint: Use induction. There are two cases: $m + 1 > |V|$ and $m + 1 \leq |V|$. In the second case, there are two subcases.)

Solution. *Proof.* (By induction.) Let $P(m)$ be the predicate as defined above.

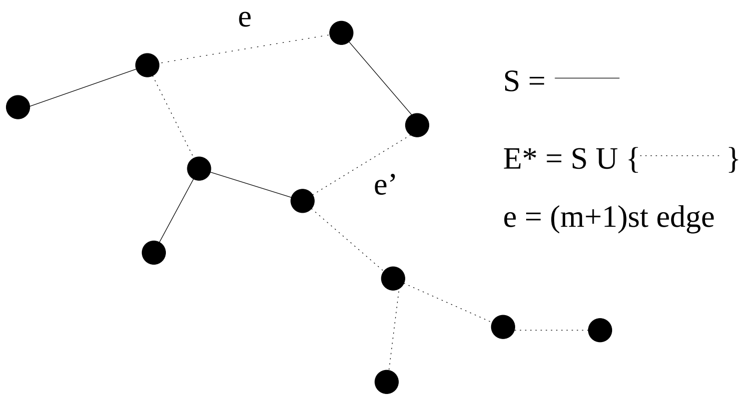
Base Case: S_0 contains 0 edges and is equal to the empty set, which is a subset of any set of edges E.

Inductive Step: Assume $P(m)$ in order to prove $P(m + 1)$.

If $m \geq |V|$ then $m+1 > |V|$ and $P(m+1)$ holds vacuously. Otherwise, if $m < |V|$ then let e denote the $(m+1)$th edge selected by ALG-GROW. By the inductive hypothesis, there exists an MST $T = (V, E)$ such that $S_m \subseteq E$. There are now two cases.

In the first case, $e \in E$ which case $S_m \cup \{e\} \subseteq E$, and thus $P(m+1)$ holds.

In the second case, $e \notin E$, as illustrated by the following diagram. Now we need to find a different MST that contains $S_m \cup \{e\}$.



What happens when we add e to T? By the description of ALG-GROW, $S_m \cup \{e\}$ does not contain a cycle. Therefore, by Lemma 4, there exists an edge $e' \neq e$ in $E - S_m$ such that $T^* = (V, E - \{e'\} \cup \{e\})$ is a spanning tree for G.

In order to prove that T^* is a MST, we need to show that $wt(e) \leq wt(e')$. We will prove this by contradiction. Suppose that $wt(e') < wt(e)$. Since $e' \in E$, which is the set of edges of the MST T, and $S_m \subseteq E$, the set of edges $S_m \cup \{e'\}$, does not contain a cycle. Therefore e' would have already been added to S_m in a previous iteration of ALG-GROW as one of the first m edges. However, e' is in $E - S_m$. This is a contradiction. ☐

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(e) Prove the theorem. (Hint: Lemma 5 says there exists an MST $T = (V, E)$ for G such that $S \subseteq E$. Use contradiction to rule out the case in which S is a proper subset of E.)

Solution. *Proof.* (by contradiction) Let S be the set of edges produced by ALG-GROW. By Lemma 5, there exists an MST $T = (V, E)$ for G such that $S \subseteq E$. If $S = E$, then ALG-GROW outputs the edges of the MST T.

We will show that the other case, $S \neq E$, leads to a contradiction. Suppose $S \neq E$. Then there exists an edge $e \in E - S$. This implies that $S \cup \{e\} \subseteq E$. Since E is the set of edges of a tree, $S \cup \{e\}$ does not contain a cycle. Therefore, e would be added to S by ALG-GROW. So $e \in S$, and this contradicts $e \in E - S$.

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Unique MST Extension

For a graph in which all edge weights are different, there exists a unique MST, which is produced by ALG-GROW. To prove this, modify the inductive hypothesis in Lemma 5 to address a unique MST, and show that uniqueness holds in the induction step due to the unique ordering of edges by weight.