

Notes for Recitation 24

Theorem 1. Let E_1, \dots, E_n be events, and let X be the number of these events that occur. Then:

$$\text{Ex}(X) = \Pr\{E_1\} + \Pr\{E_2\} + \dots + \Pr\{E_n\}$$

Theorem 2 (Markov's Inequality). Let X be a nonnegative random variable. If $c > 0$, then:

$$\Pr\{X \geq c\} \leq \frac{\text{Ex}(X)}{c}$$

Theorem 3 (Chebyshev's Inequality). For all $x > 0$ and any random variable R ,

$$\Pr\{|R - \text{Ex}(R)| \geq x\} \leq \frac{\text{Var}[R]}{x^2}$$

Theorem 4 (Union Bound). For events E_1, \dots, E_n :

$$\Pr\{E_1 \cup \dots \cup E_n\} \leq \Pr\{E_1\} + \dots + \Pr\{E_n\}$$

Theorem 5 ("Murphy's Law"). If events E_1, \dots, E_n are mutually independent and X is the number of these events that occur, then:

$$\Pr\{E_1 \cup \dots \cup E_n\} \geq 1 - e^{-\text{Ex}(X)}$$

Theorem 6 (Chernoff Bounds). Let E_1, \dots, E_n be a collection of mutually independent events, and let X be the number of these events that occur. Then:

$$\Pr\{X \geq c \text{Ex}(X)\} \leq e^{-(c \ln c - c + 1) \text{Ex}(X)} \quad \text{when } c > 1$$

1 Getting Dressed

Sometimes I forget a few items when I leave the house in the morning.

For example, here are probabilities that I forget various pieces of footwear:

left sock	0.2
right sock	0.1
left shoe	0.1
right shoe	0.3

- a. Let X be the number of these that I forget. What is $\text{Ex}(X)$?

Solution. By Theorem 1, the expected number of events that happen is the sum of the event probabilities. So,

$$\text{Ex}(X) = 0.2 + 0.1 + 0.1 + 0.3 = 0.7$$

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- b. Upper bound the probability that I forget one or more items. Make no independence assumptions.

Solution. By the Union Bound, the probability that I forget something is at most:

$$0.1 + 0.1 + 0.2 + 0.3 = 0.7$$

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- c. Use the Markov Inequality to upper bound the probability that I forget 3 or more items.

Solution.

$$\Pr \{X \geq 3\} \leq \frac{\text{Ex}(X)}{3} = \frac{7}{30}$$

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- d. Now suppose that I forget each item of footwear independently. Use Chebyshev's Inequality to upper bound the probability that I forget two or more items.

Solution. Let X_1 be the event I bring my left sock, X_2 my right sock, X_3 my left shoe, and X_4 my right shoe. Then $\text{Ex}(X_1) = .2$, $\text{Ex}(X_2) = .1$, $\text{Ex}(X_3) = .1$, and $\text{Ex}(X_4) = .3$. Moreover, since the X_i are Bernoulli random variables (binomial with $n = 1$), we have $\text{Var}[X_1] = .2(1 - .2) = .16$, $\text{Var}[X_2] = .1(1 - .1) = .09$, $\text{Var}[X_3] = .1(1 - .1) = .09$, and $\text{Var}[X_4] = .3(1 - .3) = .21$.

Let $X = \sum_{i=1}^4 X_i$. Then $\text{Ex}(X) = \sum_{i=1}^4 \text{Ex}(X_i) = .2 + .1 + .1 + .3 = .7$. Since the X_i are independent, $\text{Var}[X] = \sum_{i=1}^4 \text{Var}[X_i] = .16 + .09 + .09 + .21 = .55$. Now by Chebyshev's Inequality,

$$\begin{aligned} \Pr\{X \geq 2\} &\leq \Pr\{|X - .7| \geq 1.3\} \\ &= \Pr\{|X - \text{Ex}(X)| \geq 1.3\} \\ &\leq \frac{\text{Var}[X]}{1.3^2} \\ &= \frac{.55}{1.3^2} \\ &\leq .326 \end{aligned}$$

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e. Use Theorem 5 to lower bound the probability that I forget one or more items.

Solution. Plugging into Theorem 5, the probability that I forget one or more items:

$$1 - e^{-\text{Ex}(X)} = 1 - e^{-0.7} = 0.503\dots$$

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g. I'm supposed to remember many other items, of course: clothing, watch, backpack, notebook, pencil, kleenex, ID, keys, etc. Let X be the total number of items I remember. Suppose I remember items mutually independently and $\text{Ex}(X) = 36$. Use Chernoff's Bound to give an upper bound on the probability that I remember 48 or more items.

Solution. By the Chernoff Bound,

$$\begin{aligned} \Pr\{X \geq 48\} &= \Pr\{X \geq (1 + 1/3) \text{Ex}(X)\} \\ &\leq e^{-(4/3 \ln 4/3 - 4/3 + 1) \cdot 36} \\ &\approx .1638 \end{aligned}$$

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h. Give an upper bound on the probability that I remember 108 or more items.

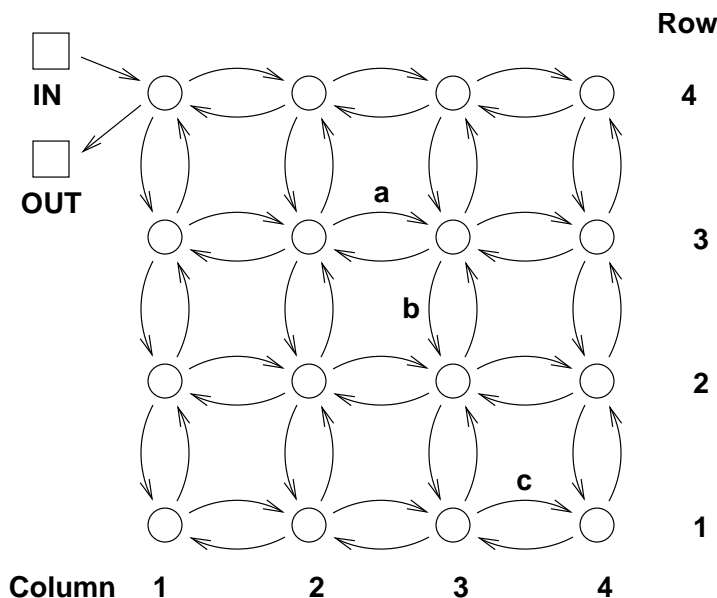
Solution. By the Chernoff Bound,

$$\begin{aligned} \Pr\{X \geq 108\} &= \Pr\{X \geq 3 \cdot \text{Ex}(X)\} \\ &\leq e^{-(3 \ln 3 - 3 + 1) \cdot 36} \\ &\leq e^{-46} \approx 1.05 \times 10^{-20}. \end{aligned}$$

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2 NxN Array

A routing network called an $n \times n$ array is shown below for $n = 4$. There is an input terminal and an output terminal attached to every node. But for clarity only one such pair of terminals is shown.



A packet travels between two nodes by first moving horizontally to the correct column and then vertically to the correct row. Suppose that each input sends one packet to an output selected uniformly and independently at random. (So zero, one, two, or more packets can be sent to a single output.) The goal of this problem is analyze congestion in an array using probability tools.

- a. What is the expected number of packets that cross edge a in the 4×4 array shown above? Also, compute the expected number of packets that cross edges b and c .

Solution. Edge a can only be crossed by packets originating at the two nodes directly to its left. Each of these packets crosses edge a if and only if its destination is in the right half of the network, which happens with probability $1/2$. Therefore, the expected number of packets crossing edge a is $2 \cdot (1/2) = 1$.

Edge b can be crossed by a packet originating at any of the 8 nodes in the upper half of the network. Each of these packets crosses edge b if and only if its destination is one of the two nodes directly below b , which happens with probability $2/16 = 1/8$. Thus, the expected number of packets crossing edge b is $8 \cdot (1/8) = 1$.

Edge c can be crossed only by a packet originating at one of the three nodes to its left. Each of these packets crosses edge c if and only if its destination is in the righthand column, which happens with probability $1/4$. Therefore, the expected number of packets that cross edge c is $3 \cdot (1/4) = 3/4$. ■

- b. Now consider an $n \times n$ array. Number the rows from 1 to n and number the columns from 1 to n . What is the expected number of packets that cross an upward edge e from row k to row $k + 1$?

Solution. A packet crosses edge e if and only if it originates in one of the bottom k rows and is destined for one of the $n - k$ nodes directly above edge e . Therefore, the expected number of packets crossing edge e is:

$$nk \cdot \frac{n - k}{n^2} = \frac{k(n - k)}{n}$$

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- c. What is the expected number of packets that cross a rightward edge f from column k to column $k + 1$?

Solution. A packet crosses edge f if and only if it originates at one of the k nodes directly left of f and is destined for one of the $n - k$ rows to the right of f . Thus, the expected number of packets that cross edge f is:

$$k \cdot \frac{(n - k)n}{n^2} = \frac{k(n - k)}{n}$$

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- d. Let μ be the expected number of packets crossing one of the edges with the greatest expected congestion. What is μ ? (Assume n is even.)

Solution. The expressions from the two preceding problem parts are both maximized when $k = n/2$. Thus, for any edge joining two different quadrants of the $n \times n$ array, the expected number of crossing packets is $n/4$. So $\mu = n/4$. ■

- e. Let X be the number of packets that cross a particular edge. We know that $\text{Ex}(X) \leq \mu$. But if we're unlucky and something weird happens, then X might be much greater, which means that the edge is especially congested. Give an upper bound on:

$$\Pr \left\{ X \geq \mu + \sqrt{2n \ln n} \right\}$$

Assume that n is large enough that the second Chernoff inequality applies. (Note that $\sqrt{2n \ln n}$ is quite small compared to μ for large n ; so we're trying to show that even small deviations above the average congestion are unlikely.)

Solution. First, we need to rewrite the probability above in a form suitable for a Chernoff bound argument:

$$\Pr \left\{ X \geq \frac{n}{4} + \sqrt{2n \ln n} \right\} = \Pr \left\{ X \geq \frac{n}{4}(1 + \delta) \right\}$$

Here $\delta = \sqrt{32 \ln n/n}$. Now we can apply the second Chernoff inequality for sufficiently large n :

$$\Pr \left\{ X \geq \frac{n}{4}(1 + \delta) \right\} \leq e^{-\delta^2 \text{Ex}(X)/3} \\ = n^{-8/3}$$

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- f. We've now shown that *one* particular edge is not likely to be congested. However, an $n \times n$ array contains a *lot* of edges. So there are a lot of different places where something could go wrong. Compute the number of edges in an $n \times n$ array and then use the Union Bound to upper bound the probability that $\mu + \sqrt{2n \ln n}$ or more packets cross some edge.

Solution. The total number of edges is $4(n-1)n$. Thus, by the Union Bound, the probability that *some* edge is crossed by more than $\mu + \sqrt{2n \ln n}$ packets is at most:

$$\frac{4(n-1)n}{n^{8/3}} \leq \frac{4}{n^{2/3}}$$

To illustrate the significance of these formulas, let's consider a huge network with $n = 10,000$. In this case, the largest number of packets expected at any edge is 2,500. By implementing edges with packet capacity $2500 + \sqrt{2 \cdot 10000 \ln 10000} = 2930$, we will have a network whose probability of overload failure is at most $4/(10000^{2/3}) < 0.01$. That is, by implementing edges that can handle a load up to only 16% higher than expected, the overall probability of overload failure anywhere in the 400 million edge network is less than 1%. ■