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Notes for Recitation 18

The Four-Step Method

This is a good approach to questions of the form, "What is the probability that ——?" Intuition *will* mislead you, but this formal approach gives the right answer every time.

1. Find the sample space. (Use a tree diagram.)
2. Define events of interest. (Mark leaves corresponding to these events.)
3. Determine outcome probabilities:
   (a) Assign edge probabilities.
   (b) Compute outcome probabilities. (Multiply along root-to-leaf paths.)
4. Compute event probabilities. (Sum the probabilities of all outcomes in the event.)

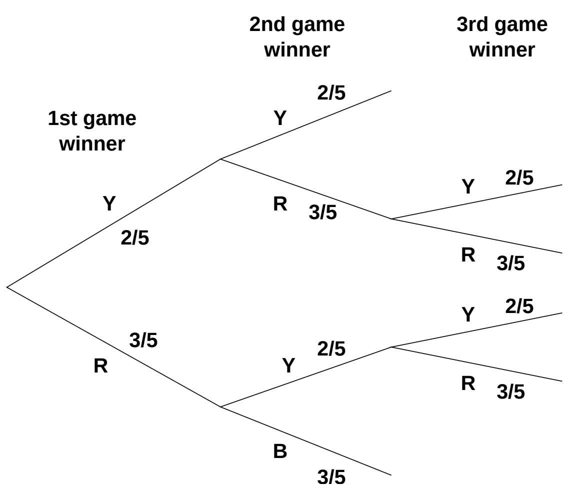
1 A Baseball Series

The New York Yankees and the Boston Red Sox are playing a two-out-of-three series. (In other words, they play until one team has won two games. Then that team is declared the overall winner and the series ends.) Assume that the Red Sox win each game with probability 3/5, regardless of the outcomes of previous games.

Answer the questions below using the four-step method. You can use the same tree diagram for all three problems.

1. What is the probability that a total of 3 games are played?
2. What is the probability that the winner of the series loses the first game?
3. What is the probability that the *correct* team wins the series?

Solution. A tree diagram is worked out below.



outcome	3 games played?	winner lost first game?	correct team wins?	Probability
YY				4/25
YRY	X			12/125
YRR	X	X	X	18/125
RYY	X	X		12/125
RYR	X		X	18/125
RR			X	9/25

From the tree diagram, we get:

$$\Pr(3 \text{ games played}) = \frac{12}{125} + \frac{18}{125} + \frac{12}{125} + \frac{18}{125} = \frac{12}{25}$$

$$\Pr(\text{winner lost first game}) = \frac{18}{125} + \frac{12}{125} = \frac{6}{25}$$

$$\Pr(\text{correct team wins}) = \frac{18}{125} + \frac{18}{125} + \frac{9}{25} = \frac{81}{125}$$

■

2 The Four-Door Deal

Suppose that *Let's Make a Deal* is played according to different rules. Now there are four doors, with a prize hidden behind one of them. The contestant is allowed to pick a door. The host must then reveal a different door that has no prize behind it. The contestant is allowed to stay with his or her original door or to pick one of the other two that are still closed. If the contestant chooses the door concealing the prize in this second stage, then he or she wins.

1. Contestant Stu, a sanitation engineer from Trenton, New Jersey, stays with his original door. What is the probability that he wins the prize?

   The tree diagram is awkwardly large. This often happens; in fact, sometimes you'll encounter *infinite* tree diagrams! Try to draw enough of the diagram so that you understand the structure of the remainder.

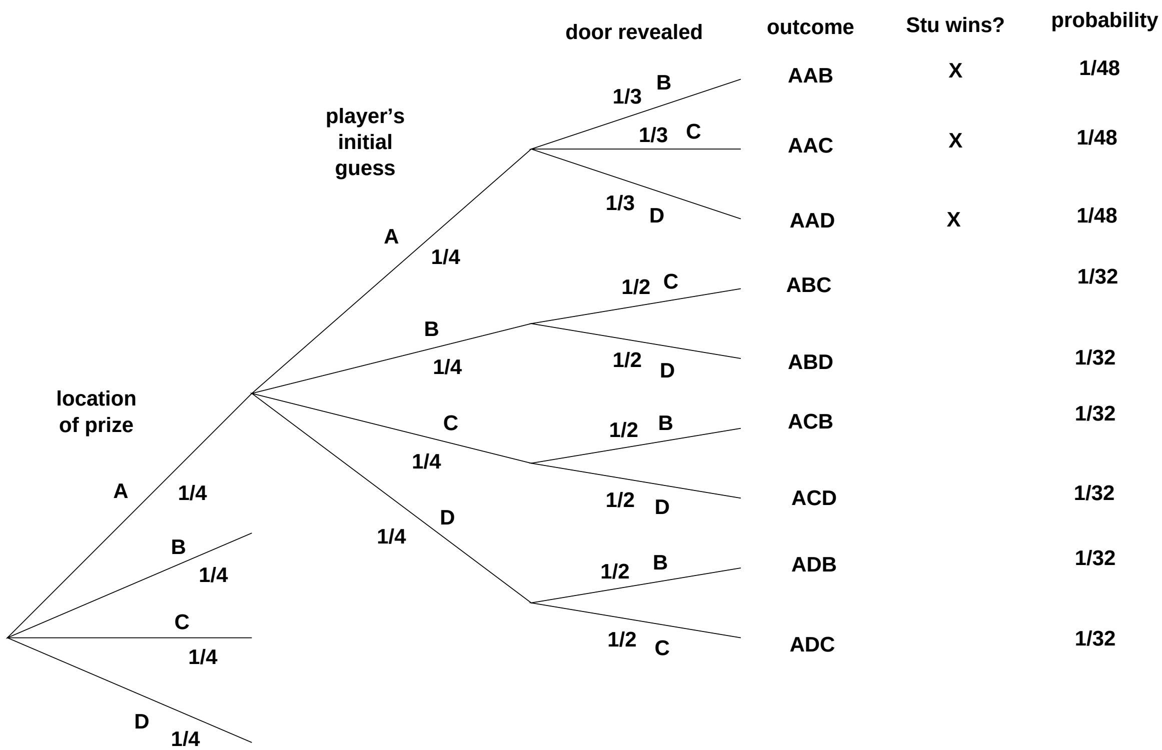
   Solution. Let's make the following assumptions:

   (a) The prize is equally likely to be behind each door.

   (b) The contestant is equally likely to pick each door initially, regardless of the prize's location.

   (c) The host is equally likely to reveal each door that does not conceal the prize and was not selected by the player.

   A partial tree diagram is shown below. The remaining subtrees are symmetric to the full-expanded subtree.



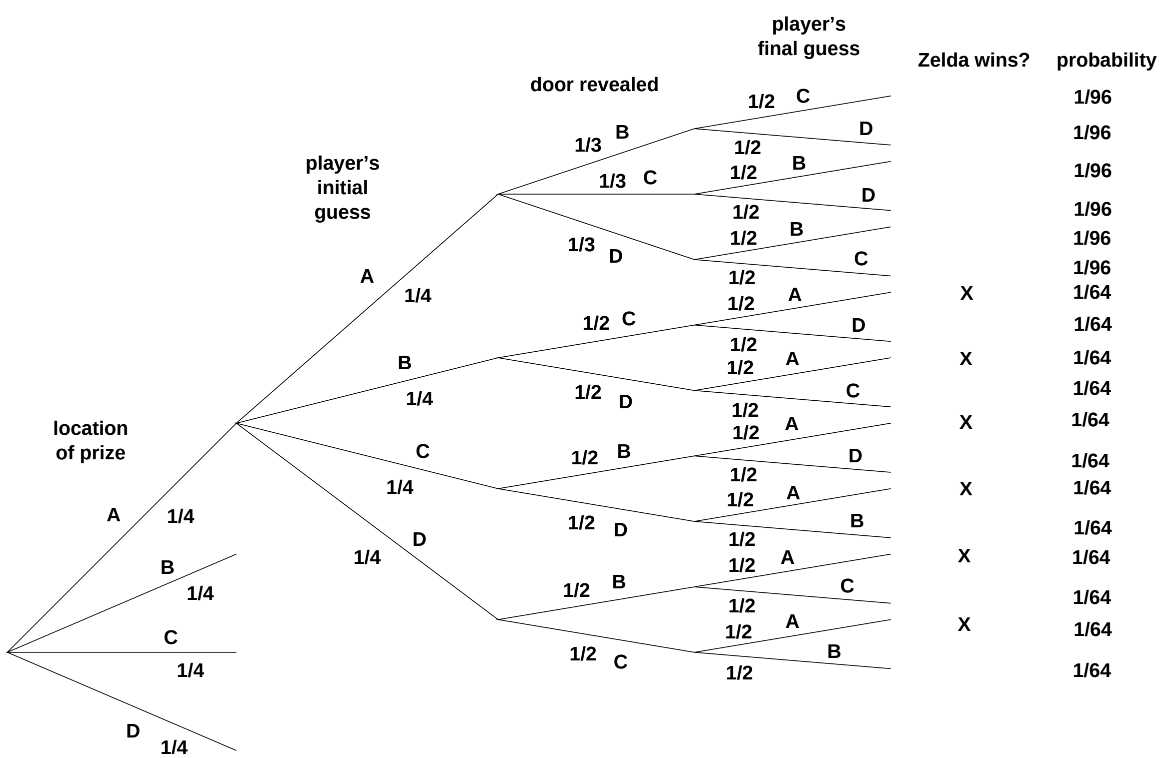
The probability that Stu wins the prize is:

$$\Pr(\text{Stu wins}) = 4 \cdot \left(\frac{1}{48} + \frac{1}{48} + \frac{1}{48} \right) = \frac{1}{4}$$

We multiply by 4 to account for the four subtrees, of which we've only drawn one. ■

2. Contestant Zelda, an alien abduction researcher from Helena, Montana, switches to one of the remaining two doors with equal probability. What is the probability that she wins the prize?

Solution. A partial tree diagram is worked out below.



The probability that Zelda wins the prize is:

$$\Pr(\text{Zelda wins}) = 4 \cdot \left(\frac{1}{64} + \frac{1}{64} + \frac{1}{64} + \frac{1}{64} + \frac{1}{64} + \frac{1}{64} \right) = \frac{3}{8}$$

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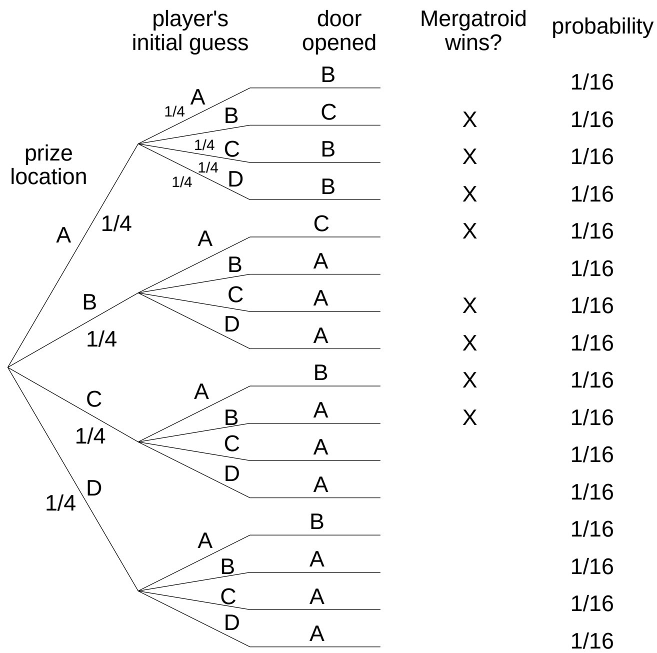
3 Earliest Door

Let's consider another variation of the four-doors problem. Say the doors are labeled A, B, C, and D. Suppose that Carol always opens the *earliest* door possible (the door whose label is earliest in the alphabet) with the restriction that she can neither reveal the prize nor open the door that the player picked.

This gives contestant Mergatroid— an engineering student from Cambridge, MA— just a little more information about the location of the prize. Suppose that Mergatroid always switches to the earliest door, excluding his initial pick and the one Carol opened. What is the probability that he wins the prize?

Solution. A tree diagram is worked out below.



The probability that Mergatroid wins is:

$$\Pr(\text{win}) = 8 \cdot \frac{1}{16} = \frac{1}{2}$$

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4 Get Even

A red die is weighted so that each even number is twice as likely to be rolled as each odd number. A blue die is similarly weighted. If both dice are rolled, what is the probability that the sum of the dice will be even?

Solution. Let x be the probability of rolling a particular odd number. Since there are three odd numbers and three even numbers, we have that $3x + 3(2x) = 1$, so $x = 1/9$. The probability that a die will come up odd when rolled is $3/9 = 1/3$, while the probability that it will come up even is $2/3$.

The sum of the dice will be even if and only if both dice come up even or both dice come up odd. By the product rule, the probability of the former is $(2/3)^2 = 4/9$, while the probability of the latter is $(1/3)^2 = 1/9$. By the sum rule, the probability that the sum of the dice is even is $4/9 + 1/9 = 5/9$. ■