Notes for Recitation 16

1 The Generalized Product Rule

Solve the following counting problems using the generalized product rule.

1. Next week, I'm going to get really fit! On day 1, I'll exercise for 5 minutes. On each subsequent day, I'll exercise 0, 1, 2, or 3 minutes more than the previous day. For example, the number of minutes that I exercise on the seven days of next week might be 5, 6, 9, 9, 9, 11, 12. How many such sequences are possible?

   Solution. The number of minutes on the first day can be selected in 1 way. The number of minutes on each subsequent day can be selected in 4 ways. Therefore, the number of exercise sequences is $1 \cdot 4^6$ by the extended product rule. ■

2. An ***r-permutation*** of a set is a sequence of r distinct elements of that set. For example, here are all the 2-permutations of $\{a, b, c, d\}$:

$$\begin{array}{ccc} (a,b) & (a,c) & (a,d) \\ (b,a) & (b,c) & (b,d) \\ (c,a) & (c,b) & (c,d) \\ (d,a) & (d,b) & (d,c) \end{array}$$

   How many r-permutations of an n-element set are there? Express your answer using factorial notation.

   Solution. There are n ways to choose the first element, $n-1$ ways to choose the second, $n-2$ ways to choose the third, ..., and there are $n-r+1$ ways to choose the r-th element. Thus, there are:

$$n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) \quad = \quad \frac{n!}{(n-r)!}$$

   r-permutations of an n-element set. ■

3. How many $n \times n$ matrices are there with *distinct* entries drawn from $\{1, \ldots, p\}$, where $p \geq n^2$?

Solution. There are p ways to choose the first entry, $p-1$ ways to choose the second, $p-2$ ways of choosing the third, and so forth. In all there are

$$p(p-1)(p-2)\cdots(p-n^2+1) = \frac{p!}{(p-n^2)!}$$

such matrices. Alternatively, this is the number of n^2-permutations of a p element set, which is $p!/(p-n^2)!$. ■

2 The Tao of BOOKKEEPER

In this problem, we seek enlightenment through contemplation of the word $BOOKKEEPER$.

1. In how many ways can you arrange the letters in the word $POKE$?

   Solution. There are 4! arrangments corresponding to the 4! permutations of the set $\{P, O, K, E\}$. ■

2. In how many ways can you arrange the letters in the word BO_1O_2K? Observe that we have subscripted the O's to make them distinct symbols.

   Solution. There are 4! arrangments corresponding to the 4! permutations of the set $\{B, O_1, O_2, K\}$. ■

3. Suppose we map arrangements of the letters in BO_1O_2K to arrangements of the letters in $BOOK$ by erasing the subscripts. Indicate with arrows how the arrangements on the left are mapped to the arrangements on the right.

O_2BO_1K	
KO_2BO_1	$BOOK$
O_1BO_2K	$OBOK$
KO_1BO_2	$KOBO$
BO_1O_2K	...
BO_2O_1K	
...	

4. What kind of mapping is this, young grasshopper?

   Solution. 2-to-1 ■

5. In light of the Division Rule, how many arrangements are there of $BOOK$?

   Solution. $4!/2$ ■

6. Very good, young master! How many arrangements are there of the letters in $KE_1E_2PE_3R$?

Solution. 6! ■

7. Suppose we map each arrangement of $KE_1E_2PE_3R$ to an arrangement of $KEEPER$ by erasing subscripts. List all the different arrangements of $KE_1E_2PE_3R$ that are mapped to $REPEEK$ in this way.

   Solution. $RE_1PE_2E_3K$, $RE_1PE_3E_2K$, $RE_2PE_1E_3K$, $RE_2PE_3E_1K$, $RE_3PE_1E_2K$, $RE_3PE_2E_1K$ ■

8. What kind of mapping is this?

   Solution. 3!-to-1 ■

9. So how many arrangements are there of the letters in $KEEPER$?

   Solution. 6!/3! ■

10. *Now you are ready to face the BOOKKEEPER!*

    How many arrangements of $BO_1O_2K_1K_2E_1E_2PE_3R$ are there?

    Solution. 10! ■

11. How many arrangements of $BOOK_1K_2E_1E_2PE_3R$ are there?

    Solution. 10!/2! ■

12. How many arrangements of $BOOKKE_1E_2PE_3R$ are there?

    Solution. $10!/(2! \cdot 2!)$ ■

13. How many arrangements of $BOOKKEEPER$ are there?

    Solution. $10!/(2! \cdot 2! \cdot 3!)$ ■

14. How many arrangements of $VOODOODOLL$ are there?

    Solution. $10!/(2! \cdot 2! \cdot 5!)$ ■

15. **(IMPORTANT)** How many n-bit sequences contain k zeros and $(n-k)$ ones?

    Solution. $n!/(k! \cdot (n-k)!)$ ■

    This quantity is denoted $\binom{n}{k}$ and read "n choose k". You will see it almost every day in 6.042 from now until the end of the term.

Remember well what you have learned: subscripts on, subscripts off.

This is the Tao of Bookkeeper.

3 More Counting Problems

Solve the following counting problems. Define an appropriate mapping (bijective or k-to-1) between a set whose size you know and the set in question.

1. **(IMPORTANT)** In how many ways can k elements be chosen from an n-element set $\{x_1, x_2, \ldots, x_n\}$?

   Solution. There is a bijection from n-bit sequences with k ones and $n - k$ zeros. The sequence $(b_1, \ldots, b_n)$ maps to the subset that contains x_i if and only if $b_i = 1$. Therefore, the number of such subsets is $\binom{n}{k}$. ■

2. How many different ways are there to select a dozen donuts if five varieties are available? (We discussed a bijection for this set in Recitation 15. Now use that bijection to give a count.)

   Solution. There is a bijection from selections of a dozen donuts to 16-bit sequences with exactly 4 ones. In particular, suppose that the varieties are glazed, chocolate, lemon, sugar, and Boston creme. Then a selection of g glazed, c chocolate, l lemon, s sugar, and b Boston creme maps to the sequence:

   $$(g\ 0's)\ 1\ (c\ 0's)\ 1\ (l\ 0's)\ 1\ (s\ 0's)\ 1\ (b\ 0's)$$

   Therefore, the number of selections is equal to the number of 16-bit sequences with exactly 4 ones, which is:

   $$\frac{16!}{4!\ 12!} = \binom{16}{4}$$

   ■

3. An independent living group is hosting eight pre-frosh, affectionately known as $P_1, \ldots, P_8$ by the permanent residents. Each pre-frosh is assigned a task: 2 must wash pots, 2 must clean the kitchen, 1 must clean the bathrooms, 1 must clean the common area, and 2 must serve dinner. In how many ways can $P_1, \ldots, P_8$ be put to productive use?

   Solution. There is a bijection from sequences containing two P's, two K's, a B, a C, and two D's. In particular, the sequence $(t_1, \ldots, t_8)$ corresponds to assigning P_i to washing pots if $t_i = P$, to cleaning the kitchen if $t_i = K$, to cleaning the bathrooms if $t_i = B$, etc. Therefore, the number of possible assignments is:

   $$\frac{8!}{2!\ 2!\ 1!\ 1!\ 2!}$$

   ■

4. Suppose that two identical 52-card decks of are mixed together. In how many ways can the cards in this double-size deck be arranged?

   Solution. The number of sequences of the 104 cards containing 2 of each card is $104!/(2!)^{52}$. ■

4 Fun with Phonology: Hawaiian

The Hawaiian language is rich in vowels: it contains 8 consonants and 25 vowels[1]! In addition, every word in Hawaiian must end in a vowel and must not contain two consonants in a row. Let's assume that all combinations of vowels and consonants that satisfy these constraints are valid.

We'd like to know how many n-phoneme words there are in Hawaiian. (A *phoneme* is either a single vowel or a single consonant. Assume no phoneme can be both a vowel and a consonant.) For simplicity, let's assume n is even.

1. Before tackling the general problem, work out how many different words there are with exactly 4 phonemes. (Which distributions of vowels and consonants are possible?)

   Solution. Since a consonant cannot go at the end of a word and no consonant can directly follow another (or equivalently, each consonant must be followed by a vowel), we have these possibilities for vowel/consonant distributions:

   $$VVVV \quad VVCV \quad VCVV \quad CVVV \quad CVCV$$

   Since these are mutually exclusive, we can find the number of words for each of the five types and sum them together. Using the product rule for each type, we find that the total number of n-phoneme words is

   $$\begin{aligned} 25^4 + 25^2 \cdot 8 \cdot 25 + 25 \cdot 8 \cdot 25^2 + 8 \cdot 25^3 + 8 \cdot 25 \cdot 8 \cdot 25 &= 25^4 + 3 \cdot 25^3 \cdot 8 + 8^2 \cdot 25^2 \\ &= 805625 \end{aligned}$$

   ■

2. Now for the general case. Let A be the set of all n-phoneme words, and let A_k be the set of all n-phoneme words with exactly k consonants. Express $|A|$ in terms of $|A_k|$ for all possible k.

   Solution. k can range from 0 to $n/2$ since every consonant is followed by a vowel. Since the set of words with k consonants and the set of words with j consonants where $j \neq k$ are disjoint, we can use the sum rule to compute $|A|$:

   $$|A| = \sum_{k=0}^{n/2} |A_k|$$

   ■

[1] Counting long vowels and diphthongs. For this problem, treat each of the 25 vowels as a unique single vowel.

3. Now let's find $|A_k|$ for an arbitrary k. For simplicity's sake, assume Hawaiian has only one consonant and only one vowel. Find a bijection between A_k and a set of arbitrary sequences of `0` and `1` of length p. What is p?

   Solution. Since every consonant must be followed by a vowel, we can group each consonant and the vowel after it into a cluster. If there are k consonants, then there are k clusters. Since there are no further constraints on the distribution of these clusters, we can map each cluster to `0` and each remaining vowel to `1`. Since the clustering removes k vowels from consideration, and the number of consonants is equal to the number of clusters, the resulting sequences of `0` and `1` have length $n - k$. ■

4. Using this bijection, compute $|A_k|$.

   Solution. The number of sequences of k `0`'s and $n - 2k$ `1`'s is

$$\binom{n-k}{k}$$

■

5. How would you change your expression for $|A_k|$ to allow for 8 consonants and 25 vowels, not just one of each?

   Solution. Each word in A_k is a sequence of V's and C's, where each V can represent any vowel and each C can represent any consonant. The total number of these sequences is $\binom{n-k}{k}$, as derived in the previous part.

   Since each sequence has k C's and $n - k$ V's, there are $8^k \cdot 25^{n-k}$ distinct words that map to the same sequence of V's and C's. In other words, this mapping is $(8^k \cdot 25^{n-k})$-to-1, so

$$|A_k| = \binom{n-k}{k} \cdot 8^k \cdot 25^{n-k}$$

■

6. How many n-phoneme words are there in Hawaiian? (You don't have to find a closed form for your expression.)

   Solution. Plugging this into the summation, we get

$$\begin{aligned} |A| &= \sum_{k=0}^{n/2} |A_k| \\ &= \sum_{k=0}^{n/2} \binom{n-k}{k} \cdot 8^k \cdot 25^{n-k} \end{aligned}$$

■

5 More Fun with Phonology: Korean (Extra)

This is just an extra problem you can work on if you have time. You shouldn't attempt this unless you've finished all the other problems. We won't be covering the solution to this in class.

In the Korean language, words are arranged in syllables where each syllable contains one initial consonant followed by one vowel. In addition, a syllable may have an optional final consonant following the vowel. There are a total of 19 initial consonants, 21 vowels, and 27 final consonants.[2] Let's assume that all combinations satisfying this constraint are valid.

1. How many 6-phoneme Korean words are there?

   Solution. Since a Korean syllable is either two phonemes long or three phonemes long, a 6-phoneme word must contain either two syllables or three syllables. These distributions are mutually exclusive.

   All words with two 3-phoneme syllables contain two initial consonants, two vowels, and two final consonants, for a total of $19^2 \cdot 21^2 \cdot 27^2$ words.

   All words with three 2-phoneme syllables contain three initial consonants and three vowels, for a total of $19^3 \cdot 21^3$ words.

   The total number of 6-phoneme words is $19^2 \cdot 21^2 \cdot 27^2 + 19^3 \cdot 21^3 = 179578728$.

   ■

2. How many n-phoneme Korean words are there?

   Solution. Suppose an n-phoneme word contains k 3-phoneme syllables. Then the word is valid only if $n - 3k$ is divisible by 2; otherwise it would be impossible to make up the remaining phonemes.

   Suppose $n - 3k$ is even. Then we have k 3-phoneme syllables and $(n - 3k)/2$ 2-phoneme syllables, for a total of $k + (n - 3k)/2 = (n - k)/2$ syllables. The number of ways to distribute these syllables is $\binom{(n-k)/2}{k}$. For each distribution, the number of possible words is $(19 \cdot 21 \cdot 27)^k (19 \cdot 21)^{(n-3k)/2} = 19^{(n-k)/2} \cdot 21^{(n-k)/2} \cdot 27^k$, so for any valid k, the total number is

   $$\binom{(n-k)/2}{k} 19^{(n-k)/2} \cdot 21^{(n-k)/2} \cdot 27^k$$

   Summing these over all possible k, we get the total number of n-phoneme words:

   $$\sum_{k=0, n-k \text{ even}}^{\lfloor n/k \rfloor} \binom{(n-k)/2}{k} 19^{(n-k)/2} \cdot 21^{(n-k)/2} \cdot 27^k$$

   ■

[2] We gloss over certain details, such as consonant clusters and diphthongs, and for simplicity, go with the convention in Korean writing (the Hangul alphabet), which separates words into syllable blocks of 2 or 3 phonemes, and treats a syllable that starts with a vowel as having a null initial consonant.

Appendix: Counting Rules

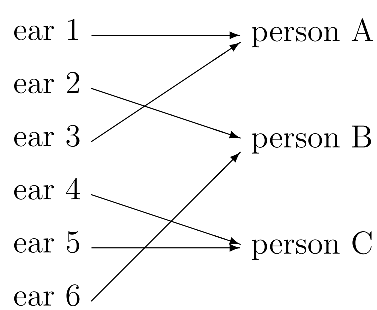
Rule 1 (Generalized Product Rule). *Let S be a set of length-k sequences. If there are:*

- *n_1 possible first entries,*
- *n_2 possible second entries for each first entry,*
- *n_3 possible third entries for each combination of first and second entries, etc.*

then:

$$|S| = n_1 \cdot n_2 \cdot n_3 \cdots n_k$$

A ***k-to-1 function*** maps exactly k elements of the domain to every element of the range. For example, the function mapping each ear to its owner is 2-to-1:



Rule 2 (Division Rule). *If $f : A \to B$ is k-to-1, then $|A| = k \cdot |B|$.*