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Notes for Recitation 12

1 Asymptotic Notation

Which of these symbols

$$\Theta \quad O \quad \Omega \quad o \quad \omega$$

can go in these boxes? (List all that apply.)

$$2n + \log n = \boxed{}(n)$$

Θ, O, Ω

$$\log n = \boxed{}(n)$$

O, o

$$\sqrt{n} = \boxed{}(\log^{300} n)$$

Ω, ω

$$n2^n = \boxed{}(n)$$

Ω, ω

$$n^7 = \boxed{}(1.01^n)$$

O, o

2 Double Sums

Sometimes we have to evaluate sums of sums, otherwise known as *double summations.* It's good to know how to tame these beasts! Here's an example of a double summation:

$$\sum_{i=1}^{n}\sum_{j=1}^{i} j$$

It looks ferocious...all those sharp teeth! But actually, this double summation is just a sheep in wolf's clothing: to evaluate it, we can just evaluate the inner sum, replace it with a closed form we already know, and then evaluate the outer sum which no longer has a summation inside it.

- Evaluate the summation. (*Hint:* $\sum(a+b) = \sum a + \sum b$.)

  Solution.

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{i} j &= \sum_{i=1}^{n} \frac{i(i+1)}{2} \\
&= 1/2 \sum_{i=1}^{n} (i^2 + i) \\
&= 1/2 (\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i) \\
&= 1/2 (\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} + \frac{n(n+1)}{2}) \\
&= 1/2 (\frac{n^3 + 3n^2 + 2n}{3})
\end{aligned}$$

■

Unfortunately, not all summations are so docile. Fortunately, we have ways to deal with this. There's a special trick that is often extremely useful for sums, and that is to *exchange the order of summation.* We'll go through an example here.

For the remainder of the problem we'll wrestle with the sum of the harmonic numbers:

$$\sum_{k=1}^{n} H_k$$

At first glance, it looks like just a single summation, but do not be deceived.

- First, write it as a double summation.

Solution.

$$\sum_{k=1}^{n} H_k = \sum_{k=1}^{n} \sum_{j=1}^{k} 1/j$$

■

- Now try to gain some intuition for exactly what you're up against by integrating the summation in its less threatening single-summation form. You may use $H_k \approx \ln k$.

Solution.

$$\sum_{k=1}^{n} H_k \approx \int_{k=1}^{n} \ln n = n \ln n - n + 1.$$

■

- Finally, we'll look for an exact answer. If we think about the pairs (k, j) over which we are summing, they form a triangle in the table below. The values in the cells of the table correspond to the terms in the double summation. The first two rows have been filled in for you. Complete the remaining three rows to see the pattern.

j	1	2	3	4	...	n
k						
1	1					
2	1	1/2				
3						
4						
	...					
n					...	

Solution.

j	1	2	3	4	...	n
k						
1	1					
2	1	1/2				
3	1	1/2	1/3			
4	1	1/2	1/3	1/4		
	...					
n	1	1/2	1/3	1/4	...	$1/n$

■

- The summation above is summing each row and then adding the row sums. But we can tame this beast if, instead, we first sum the columns and then add the column sums. Use the table to rewrite the double summation. The inner summation should sum over k, and the outer summation should sum over j.

Solution.

$$\begin{aligned}\sum_{k=1}^{n} H_k &= \sum_{k=1}^{n}\sum_{j=1}^{k} 1/j \\ &= \sum_{j=1}^{n}\sum_{k=j}^{n} 1/j\end{aligned}$$

■

- Now simplify the summation to derive a closed form in terms of n and H_n.

Solution.

$$\begin{aligned}\sum_{k=1}^{n} H_k &= \sum_{j=1}^{n}\sum_{k=j}^{n} 1/j \\ &= \sum_{j=1}^{n} 1/j \sum_{k=j}^{n} 1 \\ &= \sum_{j=1}^{n} \frac{1}{j}(n-j+1) \\ &= \sum_{j=1}^{n} \frac{n-j+1}{j} \\ &= \sum_{j=1}^{n} \frac{n+1}{j} - \sum_{j=1}^{n} \frac{j}{j} \\ &= (n+1)\sum_{j=1}^{n} \frac{1}{j} - \sum_{j=1}^{n} 1 \\ &= (n+1)H_n - n\end{aligned}$$

Notice that the exact solution is very similar in form to the approximation we generated above. ■