

## Quiz 2

YOUR NAME: \_\_\_\_\_

Circle the name of your recitation instructor:

Amy    Angelina    Arvind    Swastik

- You may use two  $8.5 \times 11$ " sheets with notes in your own handwriting on both sides, but no other reference materials. Calculators are not allowed.
- You may assume all results presented in class.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.
- The exam ends at 9:30 PM.

GOOD LUCK!

Problem	Points	Grade	Grader
1	12		
2	10		
3	10		
4	10		
5	18		
6	10		
7	20		
8	10		
Total	100		

YOUR NAME: \_\_\_\_\_

**Problem 1.** (12 points) Circle every symbol on the left that could correctly appear in the box to its right. (Thus, for each of the six parts you may circle any number of symbols.) If the base of a logarithm is unspecified, it should be understood as base 2.

(a)  $O$   $\Omega$   $\Theta$   $o$   $\omega$   $\sim$   $n = \square (100n)$

(b)  $O$   $\Omega$   $\Theta$   $o$   $\omega$   $\sim$   $\frac{1}{n^3} + \frac{1}{n} = \square \left( \frac{1}{n^2} \right)$

(c)  $O$   $\Omega$   $\Theta$   $o$   $\omega$   $\sim$   $3^n = \square (5^n)$

(d)  $O$   $\Omega$   $\Theta$   $o$   $\omega$   $\sim$   $(\lfloor \log n \rfloor)! = \square (n)$

(e)  $O$   $\Omega$   $\Theta$   $o$   $\omega$   $\sim$   $n! = \square (n^n)$

(f)  $O$   $\Omega$   $\Theta$   $o$   $\omega$   $\sim$   $10n^2 + 1000n = \square (10n^2)$

YOUR NAME: \_\_\_\_\_

**Problem 2.** (10 points) Give upper and lower bounds for the following expression which differ by at most 1. Prove that your bounds are within 1 from each other.

$$\sum_{i=1}^n i^{-6/5} + i$$

YOUR NAME: \_\_\_\_\_

**Problem 3.** (10 points) Show that if we choose 10 points inside (or on the boundary) of a square of side-length 3, there are two points in the square with distance at most  $\sqrt{2}$  from each other.

YOUR NAME: \_\_\_\_\_

**Problem 4.** (10 points) Theodore is a very well-prepared student, and quite a peculiar fellow. He always brings certain colored pens to class. In particular,

- The number of blue pens he brings is always a positive even integer.
- The number of black pens he brings is always the same as the number of blue pens.
- The number of red pens he brings is always 0, 1 or 2, and his choice is independent of the other pens he brings.
- The number of green pens he brings is always a positive odd integer, and his choice is independent of the other pens he brings.

Let  $T_n$  denote the number of different collections of  $n$  pens that can accompany Theodore. For example,  $T_7 = 2$  since there are 2 possible collections of 7 pens:

- 2 blue, 2 black, 0 red, 3 green
- 2 blue, 2 black, 2 red, 1 green

Give a closed-form generating function for the sequence  $\langle T_0, T_1, T_2, T_3, \dots \rangle$ . The answer alone is sufficient, but we can only award partial credit if you show your work.

YOUR NAME: \_\_\_\_\_

**Problem 5.** (18 points)

(a) (6 points) The beaver flu has been spreading around MIT students. Once the virus enters a student  $S$ , it is spread to 3 new (uninfected) students from  $S$  after one day, to 2 new students from  $S$  the second day, and then is cured and leaves  $S$  on the third day. For each of the first 3 days, a few Harvard students infect 5 MIT students that have never been infected before. Before the first day, no one has ever been infected.

State a recurrence, together with enough boundary conditions so that the recurrence is well-defined, describing the number of MIT students  $T(n)$  who are sick on day  $n$ , for  $n \geq 0$ . *You do not need to solve the recurrence.*

**YOUR NAME:** \_\_\_\_\_

**(b)** (6 points) The tribonacci sequence  $T(1), T(2), \dots$  is defined by the following rules:

$$T(1) = 0, T(2) = 3,$$

and  $T(i), i > 2$ , satisfies

$$T(i) = 6T(i-1) - 9T(i-2) + 12.$$

Find a closed-form expression for the  $n$ -th tribonacci number  $T(n)$ . Prove your answer is correct.

**YOUR NAME:** \_\_\_\_\_

(c) (6 points) Suppose for a sufficiently large number  $x_0$ , for all  $x \geq x_0$ , the following recurrence holds:

$$T(x) = 4T(x/2) + 32(x/4) + x^4.$$

For all  $0 \leq x < x_0$ ,  $T(x) = 1$ . Find a function  $f(x)$  for which  $T(x) = \Theta(f(x))$ .

YOUR NAME: \_\_\_\_\_

**Problem 6.** (10 points) Find a combinatorial proof of the following identity by counting the number of pairs of sets  $(X, Y) \subseteq \{1, 2, \dots, n\} \times \{1, 2, \dots, n\}$  for which  $X \subseteq Y$ .

$$3^n = \sum_{i=0}^n \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-i}{j}.$$

YOUR NAME: \_\_\_\_\_

**Problem 7.** (20 points)

- (a) (10 points) How many 30-bit binary strings are there with exactly 3 ones for which there are at least 3 zeros in between every pair of ones? Your answer can have factorials and binomial coefficients, but should otherwise be in closed form, and should not contain any variables.

YOUR NAME: \_\_\_\_\_

(b) (10 points) Recall the game of poker. There are 52 cards in a deck. Each card has a *suit* and a *value*. There are four suits:

spades      hearts      clubs      diamonds  
                  

And there are 13 values:

2, 3, 4, 5, 6, 7, 8, 9, <sup>jack</sup>J, <sup>queen</sup>Q, <sup>king</sup>K, <sup>ace</sup>A

Thus, for example,  $8\heartsuit$  is the 8 of hearts and  $A\spadesuit$  is the ace of spades. Values farther to the right in this list are considered “higher” and values to the left are “lower”.

A *Three-of-a-Kind* is a set of five cards such that three of them have the same value  $v$ , and the other two have values  $w_1$  and  $w_2$ , respectively, where  $v$ ,  $w_1$ , and  $w_2$  are all distinct. Here are a couple of examples:

$\{ 4\spadesuit, 5\diamondsuit, 4\heartsuit, 7\heartsuit, 4\clubsuit \}$   
 $\{ 8\clubsuit, Q\clubsuit, 4\heartsuit, Q\diamondsuit, Q\spadesuit \}$

How many different Three-of-a-Kinds are there? Your answer can have factorials and binomial coefficients, but should otherwise be in closed form, and should not contain any variables.

YOUR NAME: \_\_\_\_\_

**Problem 8.** (10 points) How many alphabetical strings of length 10 are there for which the letters  $A$ ,  $B$ , and  $C$  each appear at least once? Note that each position in the string can be any of the 26 letters  $A, B, C, \dots, Z$ . Your answer can have factorials and binomial coefficients, but should otherwise be in closed form, and should not contain any variables.