

Quiz 2

YOUR NAME: _____

Circle the name of your recitation instructor:

Amy Angelina Arvind Swastik

- You may use two 8.5×11 " sheets with notes in your own handwriting on both sides, but no other reference materials. Calculators are not allowed.
- You may assume all results presented in class.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.
- The exam ends at 9:30 PM.

GOOD LUCK!

Problem	Points	Grade	Grader
1	12		
2	10		
3	10		
4	10		
5	18		
6	10		
7	20		
8	10		
Total	100		

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Problem 1. (12 points) Circle every symbol on the left that could correctly appear in the box to its right. (Thus, for each of the six parts you may circle any number of symbols.) If the base of a logarithm is unspecified, it should be understood as base 2.

(a) $O \quad \Omega \quad \Theta \quad o \quad \omega \quad \sim$ $n = \square (100n)$

(b) $O \quad \Omega \quad \Theta \quad o \quad \omega \quad \sim$ $\frac{1}{n^3} + \frac{1}{n} = \square \left(\frac{1}{n^2} \right)$

(c) $O \quad \Omega \quad \Theta \quad o \quad \omega \quad \sim$ $3^n = \square (5^n)$

(d) $O \quad \Omega \quad \Theta \quad o \quad \omega \quad \sim$ $(\lfloor \log n \rfloor)! = \square (n)$

(e) $O \quad \Omega \quad \Theta \quad o \quad \omega \quad \sim$ $n! = \square (n^n)$

(f) $O \quad \Omega \quad \Theta \quad o \quad \omega \quad \sim$ $10n^2 + 1000n = \square (10n^2)$

Solution. Underlined symbols should be circled.

O Ω Θ o ω ~
 O Ω Θ o ω ~
O Ω Θ o ω ~
 O Ω Θ o ω ~
O Ω Θ o ω ~
O Ω Θ o ω ~

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Problem 2. (10 points) Give upper and lower bounds for the following expression which differ by at most 1. Prove that your bounds are within 1 from each other.

$$\sum_{i=1}^n i^{-6/5} + i$$

Solution. The sum splits into

$$\sum_{i=1}^n i^{-6/5} + \sum_{i=1}^n i.$$

From class, $\sum_{i=1}^n i = n(n+1)/2$. We use the integration method to bound $\sum_{i=1}^n i^{-6/5}$. To avoid a pole at 0, we pull out the first term from the upper bound.

$$\int_0^n (x+1)^{-6/5} dx \leq \sum_{i=1}^n i^{-6/5} \leq 1 + \int_1^n x^{-6/5} dx$$

Evaluating the integrals gives:

$$5(x+1)^{1/5} \Big|_0^n \leq \sum_{i=1}^n i^{-6/5} \leq 1 + 5x^{1/5} \Big|_1^n$$

$$5(n+1)^{1/5} - 5 \leq \sum_{i=1}^n i^{-6/5} \leq 1 + 5n^{1/5} - 5$$

Observe that the lower bound is at least $5n^{1/5} - 5$, so the upper and lower bounds differ by at most 1. It follows that

$$5n^{1/5} - 5 + \frac{n(n+1)}{2} \leq \sum_{i=1}^n i^{-6/5} + i \leq 1 + 5n^{1/5} - 5 + \frac{n(n+1)}{2}.$$

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Problem 3. (10 points) Show that if we choose 10 points inside (or on the boundary) of a square of side-length 3, there are two points in the square with distance at most $\sqrt{2}$ from each other.

Solution. Partition the square into 9 squares each of side length 1. By the pigeonhole principle, one of these 9 squares contains 2 points. The distance between these 2 points is at most $\sqrt{2}$.

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Problem 4. (10 points) Theodore is a very well-prepared student, and quite a peculiar fellow. He always brings certain colored pens to class. In particular,

- The number of blue pens he brings is always a positive even integer.
- The number of black pens he brings is always the same as the number of blue pens.
- The number of red pens he brings is always 0, 1 or 2, and his choice is independent of the other pens he brings.
- The number of green pens he brings is always a positive odd integer, and his choice is independent of the other pens he brings.

Let T_n denote the number of different collections of n pens that can accompany Theodore. For example, $T_7 = 2$ since there are 2 possible collections of 7 pens:

- 2 blue, 2 black, 0 red, 3 green
- 2 blue, 2 black, 2 red, 1 green

Give a closed-form generating function for the sequence $\langle T_0, T_1, T_2, T_3, \dots \rangle$. The answer alone is sufficient, but we can only award partial credit if you show your work.

Solution.

$$\begin{aligned}
 T(x) &= \underbrace{(x^4 + x^8 + x^{12} + \dots)}_{\text{blue and black pens}} \cdot \underbrace{(1 + x + x^2)}_{\text{red pens}} \cdot \underbrace{(x + x^3 + x^5 + \dots)}_{\text{green pens}} \\
 &= \frac{x^4}{1 - x^4} \cdot (1 + x + x^2) \cdot \frac{x}{1 - x^2} \\
 &= \frac{x^5 + x^6 + x^7}{(1 - x^4)(1 - x^2)}
 \end{aligned}$$

The second equation follows from the formula for the sum of a geometric series. The last step is a simplification, which is not required for full credit.

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Problem 5. (18 points)

(a) (6 points) The beaver flu has been spreading around MIT students. Once the virus enters a student S , it is spread to 3 new (uninfected) students from S after one day, to 2 new (uninfected) students from S the second day, and then is cured and leaves S on the third day. For each of the first 3 days, a few Harvard students infect 5 MIT students that have never been infected before. Before the first day, no one has ever been infected, and no student is ever infected by two people at the same time.

State a recurrence, together with enough boundary conditions so that the recurrence is well-defined, describing the number of *newly* infected MIT students on day n , for $n \geq 0$. You do not need to solve the recurrence.

Solution. $T(0) = 0$. $T(1) = 5$. For $n > 1$,

$$T(n) = \begin{cases} 3T(n-1) + 2T(n-2) + 5 & \text{if } n = 2 \text{ or } 3 \\ 3T(n-1) + 2T(n-2) & \text{if } n > 3 \end{cases}$$

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(b) (6 points) The tribonacci sequence $T(1), T(2), \dots$ is defined by the following rules:

$$T(1) = 0, T(2) = 3,$$

and $T(i), i > 2$, satisfies

$$T(i) = 6T(i-1) - 9T(i-2) + 12.$$

Find a closed-form expression for the n -th tribonacci number $T(n)$. Prove your answer is correct.

Solution. This is a linear recurrence with characteristic equation $x^2 - 6x + 9 = 0$. This has a double root of 3. Thus, the homogeneous solution is

$$T(n) = a_1 3^n + a_2 n 3^n.$$

This means the general solution has the form

$$T(n) = a_1 3^n + a_2 n 3^n + g(n),$$

where $g(n)$ is a particular solution. We guess a particular solution c , which is a constant. We need $c = 6c - 9c + 12$, so $4c = 12$, and $c = 3$. Thus, the general solution has the form

$$T(n) = a_1 3^n + a_2 n 3^n + 3.$$

Using the boundary conditions, $T(1) = 0 = 3a_1 + 3a_2 + 3$ and $T(2) = 3 = 9a_1 + 18a_2 + 3$. This gives $a_1 + a_2 = -1$ and $a_1 + 2a_2 = 0$. Thus, $a_1 = -2$ and $a_2 = 1$. Therefore,

$$T(n) = -2 \cdot 3^n + n 3^n + 3 = (n-2)3^n + 3.$$

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(c) (6 points) Suppose for a sufficiently large number x_0 , for all $x \geq x_0$, the following recurrence holds:

$$T(x) = 4T(x/2) + 32T(x/4) + x^4.$$

For all $0 \leq x < x_0$, $T(x) = 1$. Find a solution to this recurrence. You can express your answer using theta notation.

Solution. We apply the Akra-Bazzi Method. Here $k = 2$, $a_1 = 4$, $a_2 = 32$, $b_1 = 1/2$, and $b_2 = 1/4$. If $g(x) = x^4$, then $g'(x) = 4x^3 = O(x^3)$. Thus, since x_0 is sufficiently large, the method applies. We solve for p , which is defined by the equation:

$$4 \left(\frac{1}{2}\right)^p + 32 \left(\frac{1}{4}\right)^p = 1.$$

The answer is $p = 3$. We compute the solution to the recurrence by integrating:

$$\begin{aligned} T(x) &= \Theta \left(x^3 \left(1 + \int_1^x \frac{u^4}{u^4} du \right) \right) \\ &= \Theta \left(x^3 \left(1 + \int_1^x 1 du \right) \right) \\ &= \Theta (x^3 (1 + x - 1)) \\ &= \Theta(x^4) \end{aligned}$$

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Problem 6. (10 points) Find a combinatorial proof of the following identity by counting the number of pairs of sets (X, Y) such that $X \subseteq Y \subseteq \{1, 2, \dots, n\}$:

$$3^n = \sum_{i=0}^n \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-i}{j}.$$

Solution. We count the number of pairs of sets $(X, Y) \subseteq \{1, 2, \dots, n\} \times \{1, 2, \dots, n\}$ for which $X \subseteq Y$ in two different ways. For each integer $i \in \{1, 2, \dots, n\}$, we can either exclude it from both X and Y , exclude it from X but include it in Y , or include it in both X and Y . Thus, the number of pairs of subsets is 3^n .

On the other hand, consider any of the $\binom{n}{i}$ sets $X \subseteq \{1, 2, \dots, n\}$ of size i , for some $0 \leq i \leq n$. The number of sets $Y \subseteq \{1, 2, \dots, n\}$ containing X is $\sum_{j=0}^{n-i} \binom{n-i}{j} = 2^{n-i}$. Indeed, for any such Y , we must include the i elements of X in Y , and there are $n - i$ elements in $\{1, 2, \dots, n\} \setminus X$, for which we can include any combination of j of them in Y , for any $0 \leq j \leq n - i$. Thus, taking the sum of $\binom{n}{i} \binom{n-i}{j}$ over all $i \leq n$ and $j \leq n - i$ gives the total number of pairs of sets $(X, Y) \subseteq \{1, 2, \dots, n\} \times \{1, 2, \dots, n\}$ for which $X \subseteq Y$.

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Problem 7. (20 points)

(a) (10 points) How many 30-bit binary strings are there with exactly 3 ones for which there are at least 3 zeros in between every pair of ones? Your answer can have factorials and binomial coefficients, but should otherwise be in closed form, and should not contain any variables.

Solution. We can set up a bijection between such strings and strings of length 24 containing exactly 3 ones. Given a string of the latter form, we may replace each 0 by a 0, the first two 1s by a 1000, and the last 1 by a 1. It is easy to see that this is a bijection, and thus there are $\binom{24}{3}$ such strings.

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(b) (10 points) Recall the game of poker. There are 52 cards in a deck. Each card has a *suit* and a *value*. There are four suits:

spades hearts clubs diamonds
   

And there are 13 values:

2, 3, 4, 5, 6, 7, 8, 9, $\overset{\text{jack}}{J}$, $\overset{\text{queen}}{Q}$, $\overset{\text{king}}{K}$, $\overset{\text{ace}}{A}$

Thus, for example, $8\heartsuit$ is the 8 of hearts and $A\spadesuit$ is the ace of spades. Values farther to the right in this list are considered “higher” and values to the left are “lower”.

A *Three-of-a-Kind* is a set of five cards such that three of them have the same value v , and the other two have values w_1 and w_2 , respectively, where v , w_1 , and w_2 are all distinct. Here are a couple of examples:

$\{ 4\spadesuit, 5\diamondsuit, 4\heartsuit, 7\heartsuit, 4\clubsuit \}$
 $\{ 8\clubsuit, Q\clubsuit, 4\heartsuit, Q\diamondsuit, Q\spadesuit \}$

How many different Three-of-a-Kinds are there? Your answer can have factorials and binomial coefficients, but should otherwise be in closed form, and should not contain any variables.

Solution. We can choose the value of the three cards with the same value in 13 ways, and then we can choose the values of the remaining two cards in $\binom{12}{2}$ ways. We can choose the three suits for the three cards with the same value in $\binom{4}{3}$ ways. We can choose the suits of the remaining two cards in $4 \cdot 4$ ways. By the generalized product rule, the number of Three-of-a-Kinds is

$$13 \cdot \binom{12}{2} \cdot \binom{4}{3} \cdot 4^2.$$

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Problem 8. (10 points) How many alphabetical strings of length 10 are there for which the letters A , B , and C each appear at least once? Note that each position in the string can be any of the 26 letters A, B, C, \dots, Z . Your answer can have factorials and binomial coefficients, but should otherwise be in closed form, and should not contain any variables.

Solution. Let A be the set of strings not containing the letter A , and similarly define sets B and C . Our answer is $26^{10} - |A \cup B \cup C|$. By the inclusion-exclusion principle,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Computing, $|A| = |B| = |C| = 25^{10}$. $|A \cap B| = |A \cap C| = |B \cap C| = 24^{10}$. $|A \cap B \cap C| = 23^{10}$.

$$|A \cup B \cup C| = 3 \cdot 25^{10} - 3 \cdot 24^{10} + 23^{10}.$$

Thus, the solution is

$$26^{10} - 3 \cdot 25^{10} + 3 \cdot 24^{10} - 23^{10}.$$