Quiz 1

Circle the name of your recitation instructor:

Amy Angelina Arvind Swastik

- This quiz is **closed book**, but you may have one $8.5 \times 11''$ sheet with notes in your own handwriting on both sides.
- Calculators are not allowed.
- You may assume all of the results presented in class.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.
- If you get stuck on a problem, move on to others. The problems are not arranged in order of difficulty.
- The exam ends at 9:30 PM.
- For this quiz, $\mathbb{N}$ is the set of nonnegative integers (including 0): $\mathbb{N} = \{0, 1, \ldots, \}$.
- GOOD LUCK!

Problem	Points	Grade	Grader
1	12		
2	16		
3	20		
4	20		
5	12		
6	10		
7	10		
Total	100		

Problem 1. [12 points] Let X be the set of students in 6.042. Let Y be the set of problems on this quiz. For $x \in X$ and $y \in Y$, let $P(x, y)$ be the statement "Student x got full points on problem y". Let $Q(x)$ be the statement "Student x drops 6.042".

(a) Convert the following statements into English.

1. $(\exists x \in X, Q(x)) \Rightarrow (\forall x \in X, Q(x))$.
2. $\forall x \in X((\exists y \in Y \neg P(x, y)) \Rightarrow Q(x))$
3. $\exists x \in X(\neg Q(x))$.

(b) Assuming 1,2 and 3 are true, what can you say about your score on this quiz?

Problem 2. [16 points] Define the Fibonacci numbers F_n as follows: $F_0 = 1$, $F_1 = 1$, $\forall n \in \mathbb{N}, F_{n+2} = F_{n+1} + F_n$.

Prove that $\forall n \in \mathbb{N}$, $F_n \geq \frac{1}{2}(1.5)^n$.

Problem 3. [20 points] 6 people are sitting in a circle. Each person has a number. They do a ritual during each round that makes their numbers update. Each person, to get his number for round $n+1$, takes his number from round n, then adds his right neighbor's number from round n to it, and then subtracts his left neighbor's number from round n from it.

For example if on the current round there is a person A whose number is 5, and A's right neighbor's number is 4, and A's left neighbor's number is 6, then in the following round, A's number will be $5+4-6=3$.

Initially the people are given the numbers $1, 2, 3, 4, 5, 6$ (in clockwise order, and the person with 6 is sitting next to the person with 1).

Thus, after 1 round, their numbers will be $5, 0, 1, 2, 3, 10$.

Prove that there will never be a round when all their numbers are equal.

BIG HINT: Consider what happens to the sum of the numbers.

Assume for the purposes of contradiction that there is a round in which all numbers are equal to the value v. By the lemma, $v = 21/6$ and v is an integer, which is impossible since 6 does not divide 21. Thus, there cannot be a round in which all numbers are equal.

Problem 4. [20 points] Let a and b be two distinct positive integers. Define a graph G as follows. The set of vertices of G is the set of *all integers*. There is an edge between n and m if and only if $((|n - m| = a)$ OR $(|n - m| = b))$.

(a) For each integer n, determine the degree of vertex n.

(b) Describe the integers n for which there is a path from vertex 0 to vertex n.

(c) Prove your answer in part (b) is correct (you may assume any result from class).

(d) Prove that G is connected if and only if a and b are relatively prime.

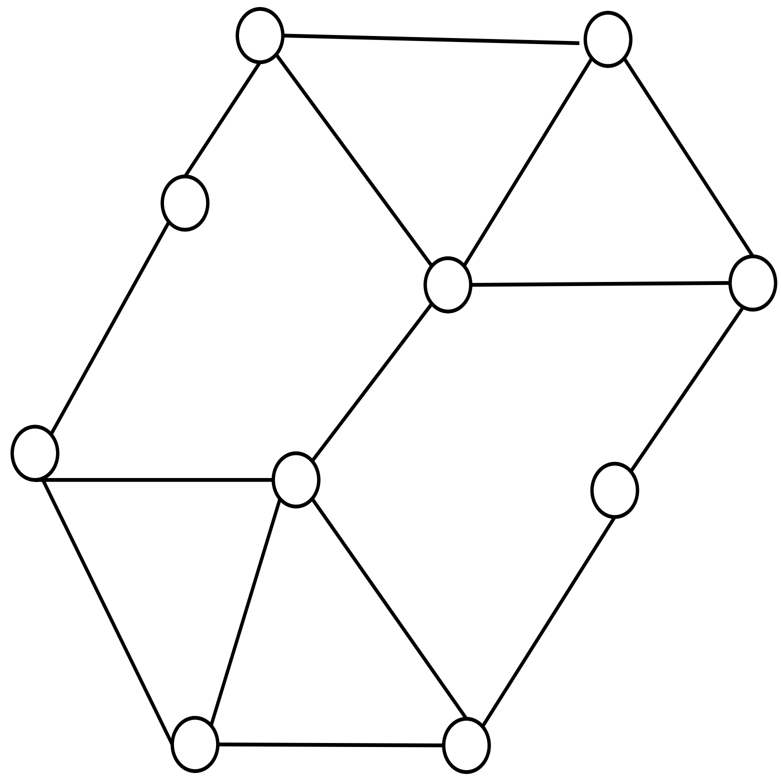

Figure 1: Graph for Problem 5

Problem 5. [12 points]

Consider the graph in figure 1. Solve the following problems: you needn't prove that your answer is correct, but if you show your reasoning it could get you partial credit in case the answer is wrong.

1. Find its diameter.

2. Find its chromatic number.

3. Show an optimal coloring of the graph (by placing a number next to each node in the graph).

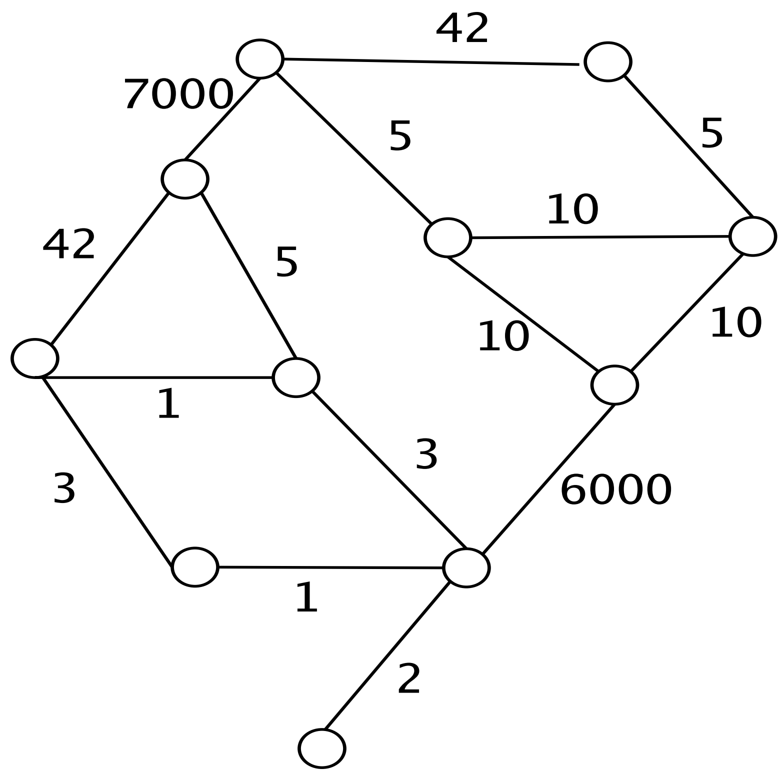

Figure 2: Graph for Problem 6

Problem 6. [10 points] Show a minimum weight spanning tree for the weighted graph shown in Figure 2 (by placing a circle on the weight of each edge in the tree). What is the minimum spanning tree's weight?

Problem 7. [10 points] Consider the following relation on the set of natural numbers:

$$R = \{(x, y) : x \leq y^2 \text{ for } x, y \in \mathbb{N}\}.$$

Which of the following properties holds for R? If it has the property, prove it. If not, provide a counterexample.

1. reflexive.

2. symmetric.

3. transitive.

4. antisymmetric.

5. equivalence relation.