

Quiz 1

Problem 1. [12 points] Let X be the set of students in 6.042. Let Y be the set of problems on this quiz. For $x \in X$ and $y \in Y$, let $P(x, y)$ be the statement “Student x got full points on problem y ”. Let $Q(x)$ be the statement “Student x drops 6.042”.

(a) Convert the following statements into English.

1. $(\exists x \in X, Q(x)) \Rightarrow (\forall x \in X, Q(x))$.

Solution. If a student in 6.042 drops the class, then all the students in 6.042 will drop the class.

2. $\forall x \in X((\exists y \in Y \neg P(x, y)) \Rightarrow Q(x))$

Solution. For every student in 6.042, if there is a problem on the quiz that he / she doesn't get full points on, he / she will drop the class.

3. $\exists x \in X(\neg Q(x))$.

Solution. There is a student in 6.042 who won't drop the class.

(b) Assuming 1,2 and 3 are true, what can you say about your score on this quiz?

Solution. I will get full points on the quiz.

Problem 2. [16 points] Define the Fibonacci numbers F_n as follows: $F_0 = 1$, $F_1 = 1$, $\forall n \in \mathbb{N}$, $F_{n+2} = F_{n+1} + F_n$.

Prove that $\forall n \in \mathbb{N}$, $F_n \geq \frac{1}{2}(1.5)^n$.

Solution.

We prove this by strong induction. Our inductive hypothesis is $P(n)$, which states that $F_n \geq \frac{1}{2}(1.5)^n$.

We have two base cases, $P(0)$ and $P(1)$. Note that $F_0 = 1 \geq \frac{1}{2}$, and $F_1 = 1 \geq \frac{1}{2} \cdot 1.5 = \frac{3}{4}$. So $P(0)$ and $P(1)$ hold.

Assume, inductively, that $P(0), P(1), \dots, P(n)$ hold, and let's show that $P(n+1)$ holds. Then, using $P(n-1)$ and $P(n)$, we have

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1} \\ &\geq \frac{1}{2}(1.5)^n + \frac{1}{2}(1.5)^{n-1} \\ &= \frac{1}{2}(1.5)^{n-1}(1.5 + 1) \\ &= \frac{1}{2}(1.5)^{n-1}\left(\frac{5}{2}\right) \\ &> \frac{1}{2}(1.5)^{n-1}\left(\frac{9}{4}\right) \\ &= \frac{1}{2}(1.5)^{n+1}, \end{aligned}$$

and so $P(n+1)$ holds by the principle of strong induction.

Problem 3. [20 points] 6 people are sitting in a circle. Each person has a number. They do a ritual during each round that makes their numbers update. Each person, to get his number for round $n + 1$, takes his number from round n , then adds his right neighbor's number from round n to it, and then subtracts his left neighbor's number from round n from it.

For example if on the current round there is a person A whose number is 5, and A 's right neighbor's number is 4, and A 's left neighbor's number is 6, then in the following round, A 's number will be $5 + 4 - 6 = 3$.

Initially the people are given the numbers 1, 2, 3, 4, 5, 6 (in clockwise order, and the person with 6 is sitting next to the person with 1).

Thus, after 1 round, their numbers will be 5, 0, 1, 2, 3, 10.

Prove that there will never be a round when all their numbers are equal.

BIG HINT: Consider what happens to the sum of the numbers.

Solution.

Lemma 1. *After every round, each number is an integer, and the sum of the numbers is 21.*

Proof. The inductive hypothesis $P(n)$ states that after round n , the sum of the numbers is 21 and all numbers are integers.

The base case $P(0)$ follows from summing up the values of the initial settings of the numbers and noting that all initial values are integers.

Assuming that $P(n)$ holds, we show that $P(n + 1)$ holds. Let the values of the numbers at round n be a_1, \dots, a_6 . For each i , the new value of the i^{th} person is $a_i + a_{i+1 \bmod 6} - a_{i-1 \bmod 6}$. Since the inductive hypothesis tells us that $a_i, a_{i+1 \bmod 6}$ and $a_{i-1 \bmod 6}$ are all integers, the new value of the i^{th} person is also an integer. The sum at round $n + 1$ is

$$\sum_{1 \leq i \leq 6} a_i + a_{i+1 \bmod 6} - a_{i-1 \bmod 6} = \sum_{1 \leq i \leq 6} a_i + a_i - a_i = \sum_{1 \leq i \leq 6} a_i = 21.$$

The second equality is by reordering the summation, and the third equality is from the inductive hypothesis. And so $P(n + 1)$ holds by the principle of induction. \square

Assume for the purposes of contradiction that there is a round in which all numbers are equal to the value v . By the lemma, $v = 21/6$ and v is an integer, which is impossible since 6 does not divide 21. Thus, there cannot be a round in which all numbers are equal.

Problem 4. [20 points] Let a and b be two distinct positive integers. Define a graph G as follows. The set of vertices of G is the set of *all integers*. There is an edge between n and m if and only if $(|n - m| = a)$ OR $(|n - m| = b)$.

(a) For each integer n , determine the degree of vertex n .

Solution. 4. Vertex n is connected to vertices $n - a$, $n + a$, $n + b$, and $n - b$, which are distinct because a and b are distinct positive integers. Call the corresponding edges the “ a -edge”, the “ $-a$ -edge”, the “ b -edge” and the “ $-b$ -edge”.

(b) Describe the integers n for which there is a path from vertex 0 to vertex n .

Solution. There is a path from vertex 0 to vertex n if and only if n is a linear combination of a and b .

(c) Prove your answer in part (b) is correct (you may assume any result from class).

Solution. We first show that if there is a path from vertex 0 to vertex n , then n is a linear combination of a and b . We will use induction on the length of the path, and our induction hypothesis $P(l)$ is that if there is a path from vertex 0 to vertex n of length l , then n is a linear combination of a and b . The base case, $P(0)$ is true, since only 0 can be reached by a path of length 0 from vertex 0, and 0 is a linear combination of a and b . Assume $P(l)$ is true, we show that $P(l + 1)$ holds: Let vertex n be reachable from 0 by a path $0, x_1, x_2, \dots, x_l, x_{l+1} = n$ of length $l + 1$. Since x_l can be reached by a path of length l from 0, it is a linear combination of a and b , i.e. there exist integers s and t such that $x_l = sa + tb$. Since, x_l is adjacent to n , by definition of the graph, $n \in \{x_l + a, x_l - a, x_l + b, x_l - b\}$. Thus, $x_l \in \{(s + 1)a + tb, (s - 1)a + tb, sa + (t + 1)b, sa + (t - 1)b\}$, and so x_l is a linear combination of a and b .

If n is a linear combination of a and b , then there are integers s, t such that $n = sa + tb$. The path from 0 to n is constructed as follows: Start at 0. If s is positive, then take s steps following the a -edge of the current node, otherwise (s is negative) take s steps following the $-a$ -edge of the current node. Then, if t is positive, take t steps following the b -edge of the current node, otherwise (t is negative) take s steps following the $-b$ -edge of the current node.

(d) Prove that G is connected if and only if a and b are relatively prime.

Solution. In lecture, we showed that every linear combination of two positive integers a and b is a multiple of $\gcd(a, b)$ and vice versa. Since a path only exists from 0 to n if and only if n is a linear combination of a and b (proved in part c), a path exists from 0 to n if and only if n is a multiple of $\gcd(a, b)$. If a and b are relatively prime, then $\gcd(a, b) = 1$. If $\gcd(a, b) = 1$, then, since every n is a multiple of 1, there is a path from 0 to every vertex n . If a and b are not relatively prime then $\gcd(a, b)$ equals some other integer $k > 1$. All the vertices that are not multiples of k will be unreachable from vertex 0, and so the graph will not be connected.

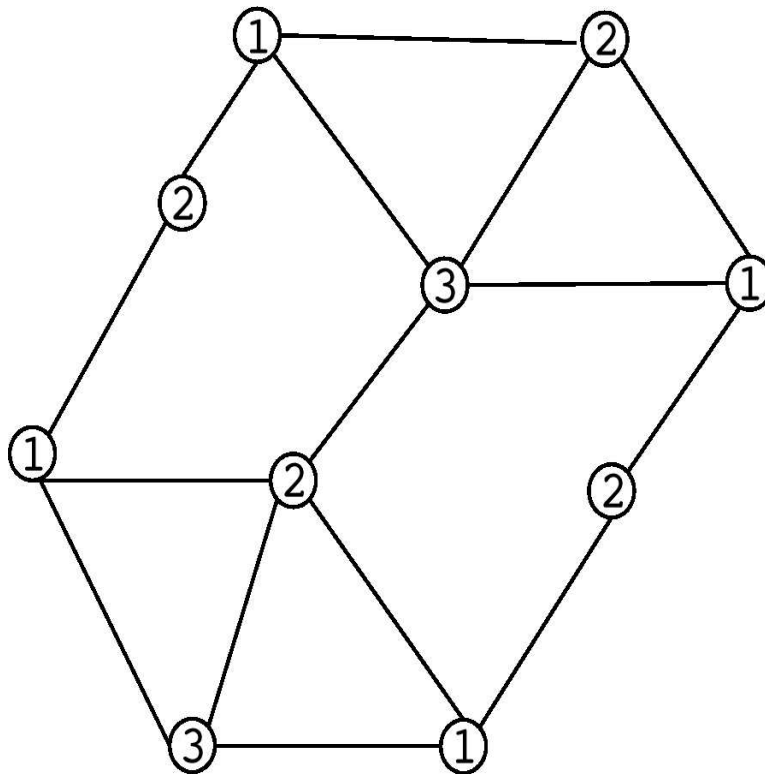


Figure 1: Graph for Problem 5

Problem 5. [12 points]

Consider the graph in figure 1. Solve the following problems: you needn't prove that your answer is correct, but if you show your reasoning it could get you partial credit in case the answer is wrong.

1. Find its diameter.

Solution. The *diameter* of a graph is the largest *distance* between any two vertices (where the *distance* between two vertices is the length of the shortest path between them). The diameter in this case is 4.

2. Find its chromatic number.

Solution. The chromatic number in this case is 3. We know it must be at least 3, because there is a complete subgraph with 3 vertices. The next part of question demonstrates that it can indeed be colored using 3 colors.

3. Show an optimal coloring of the graph (by placing a number next to each node in the graph).

Solution. See figure above. This is one solution. There are other variations.

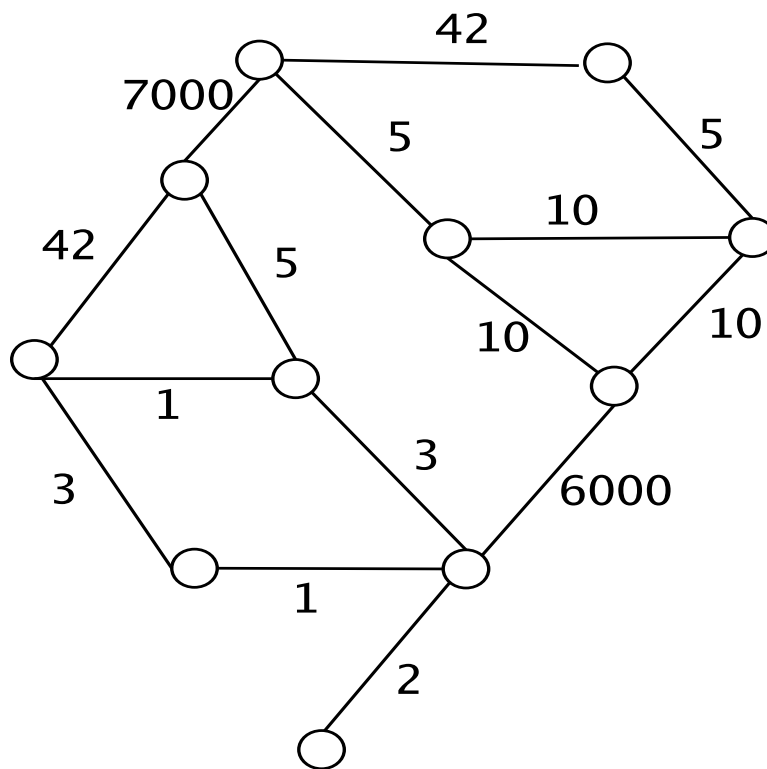


Figure 2: Graph for Problem 6

Problem 6. [10 points] Show a minimum weight spanning tree for the weighted graph shown in Figure 2 (by placing a circle on the weight of each edge in the tree). What is the minimum spanning tree's weight?

Solution.

There are several possible choices of minimum-weight spanning tree; however, none of them will contain the edge of weight 7000, nor either edge of weight 42, nor all three edges of weight 10 (it will contain two of these), nor both edges of weight 3 (it will contain only one of these). The weight of such a tree will be

$$1 + 1 + 2 + 3 + 5 + 5 + 5 + 10 + 10 + 6000 = 6042.$$

Problem 7. [10 points] Consider the following relation on the set of natural numbers:

$$R = \{(x, y) : x \leq y^2 \text{ for } x, y \in \mathbb{N}\}.$$

Which of the following properties holds for R ? If it has the property, prove it. If not, provide a counterexample.

1. reflexive.

Solution. Yes. R is reflexive if $\forall x \ xRx$, that is, if $x \leq x^2$. If $x = 0$, $0 \leq 0$, so reflexivity holds. If $x \geq 1$, then it follows that $x^2 \geq x$, from multiplying both sides of the inequality by x , which is positive. This covers all $x \in \mathbb{N}$.

2. symmetric.

Solution. No. Counterexample: $x = 0, y = 1$. $0 \leq 1^2$, but $1 > 0^2$.

3. transitive.

Solution. No. Counterexample: $x = 10, y = 5, z = 3$. $10 \leq 5^2$, and $5 \leq 3^2$, but $10 \not\leq 3^2$.

4. antisymmetric.

Solution. No. Counterexample: $x = 2, y = 3$. $2 \leq 3^2$, and $3 \leq 2^2$, but $3 \neq 2$.

5. equivalence relation.

Solution. No. An equivalence relation must be reflexive, symmetric, and transitive. Of those three, the relation is only reflexive.