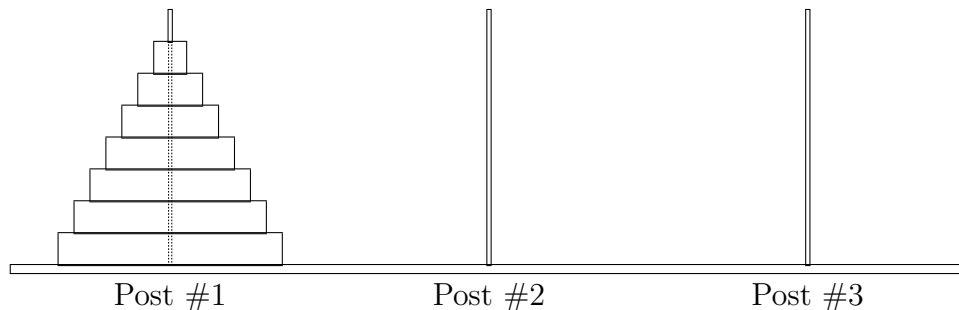


Problem Set 8

Due: Monday, October 27

Problem 1. [15 points] Less well-known than the Towers of Hanoi— but no less fascinating—are Towers of Sheboygan, WI. As in Hanoi, the puzzle in Sheboygan involves 3 posts and n disks of different sizes. Initially, all the disks are on post #1:



The objective is to transfer all n disks to post #2 via a sequence of moves. A move consists of removing the top disk from one post and dropping it onto another post with the restriction that a larger disk can never lie above a smaller disk. Furthermore, a local ordinance requires that *a disk can be moved only from post #1 to post #2, from #2 to #3, or from #3 to #1*. Thus, for example, moving a disk directly from post #1 to post #3 is not permitted.

- (a) [5 pts] Describe a solution to the Towers of Sheboygan puzzle.
- (b) [5 pts] Let S_n be the number of moves needed to solve the n -disk problem. Express S_n with a recurrence equation and sufficient base cases.
- (c) [5 pts] Find a closed-form expression for S_n by solving the recurrence.

Problem 2. [30 points] Find Θ bounds for the following divide-and-conquer recurrences, which give expressions for $T(n)$ for all integers $n \geq 2$. Assume that $T(0) = T(1) = 1$ in all cases. Show your work.

(a) [6 pts] $T(n) = 3T(\lfloor n/9 \rfloor + 1) + \sqrt{n}$

(b) [6 pts] $T(n) = 7T(\lfloor n/2 \rfloor) + n^2$

(c) [6 pts] $T(n) = 36T(\lfloor n/6 + \sqrt{n} \rfloor) + n^2$

(d) [6 pts] $T(n) = T(\lfloor n/3 \rfloor) + T(\lfloor n/12 \rfloor) + 15n$

(e) [6 pts] $T(n) = 5T(\lfloor n/8 \rfloor) + n$

Problem 3. [15 points] Define the sequence of numbers A_i by

$$A_0 = 2$$

$$A_{n+1} = A_n/2 + 1/A_n \quad (\text{for } n \geq 1)$$

Prove that $A_n \leq \sqrt{2} + 1/2^n$ for all $n \geq 0$.

Problem 4. [30 points] Find closed-form solutions to the following linear recurrences.

(a) [15 pts] $x_n = -4x_{n-1} + 3x_{n-2} + 18x_{n-3} \quad (x_0 = 2, x_1 = 2, x_2 = 3)$

Hint: -3 is a root.

(b) [15 pts] $x_n = 3x_{n-1} - 2x_{n-2} + n \quad (x_0 = 0, x_1 = 1)$

Problem 5. [10 points] The following recurrence was invented by a late Italian renaissance scholar named Giardi Invertabinacci, who earned a certain repute in his age for mocking profound ideas, poetry, sculpture, and even architecture by constructing upside-down or backward replicas of the original. His scaled model of the famed Tower of Pisa still stands—upside down, but not leaning—outside Palermo. His one venture into mathematics was this recurrence relation, recorded in a letter to a contemporary mathematician and reproduced below in modern notation.¹

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 1 \\ x_{n+1} &= \frac{x_n x_{n-1}}{x_n + x_{n-1}} \end{aligned}$$

(a) [5 pts] Guess the solution to the recurrence. Your solution may be expressed in terms of Fibonacci numbers.

(b) [5 pts] Use strong induction to prove your guess correct.

¹Yeah, yeah, we made all this up. But good story, eh?