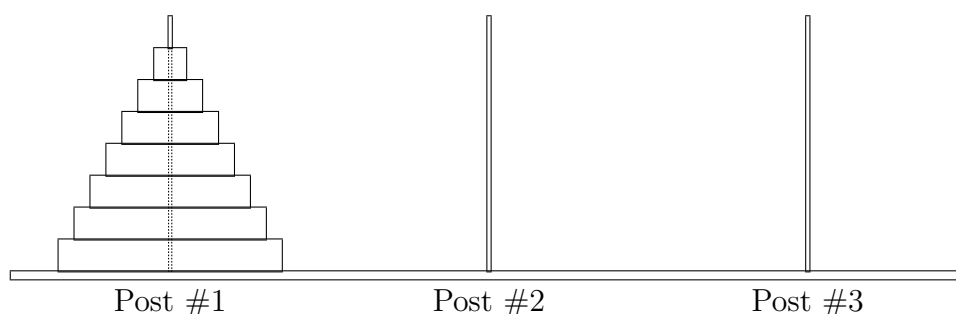


## Problem Set 8 Solutions

Due: Monday, October 27

**Problem 1.** [15 points] Less well-known than the Towers of Hanoi— but no less fascinating—are Towers of Sheboygan, WI. As in Hanoi, the puzzle in Sheboygan involves 3 posts and  $n$  disks of different sizes. Initially, all the disks are on post #1:



The objective is to transfer all  $n$  disks to post #2 via a sequence of moves. A move consists of removing the top disk from one post and dropping it onto another post with the restriction that a larger disk can never lie above a smaller disk. Furthermore, a local ordinance requires that *a disk can be moved only from post #1 to post #2, from #2 to #3, or from #3 to #1*. Thus, for example, moving a disk directly from post #1 to post #3 is not permitted.

(a) [5 pts] Describe a solution to the Towers of Sheboygan puzzle.

**Solution.** Use a recursive procedure: to move an initial stack of  $n$  blocks to the next post, move the top stack of  $n - 1$  disks to the furthest post by moving it to the next post two times, then move the big,  $n$ th disk to the next post, and finally move the top stack another two times to land on top of the big disk.

This procedure leads to a simple linear recurrence, and gets full credit as an answer. But it turns out not to be the most efficient procedure for moving the stack.

Namely, a better (indeed optimal, but we won't prove this) procedure can be defined in terms of two mutually recursive procedures, procedure  $P_1(n)$  for moving a stack of  $n$  disks 1 pole forward, and  $P_2(n)$  for moving a stack of  $n$  disks 2 poles forward. It's obvious how to do this for  $n = 1$ . For  $n > 1$ , define:

$P_1(n)$ : Apply  $P_2(n - 1)$  to move the top  $n - 1$  disks two poles forward to the third pole. Then move the remaining big disk once to land on the second pole. Then apply  $P_2(n - 1)$

again to move the stack of  $n - 1$  disks two poles forward from the third pole to land on top of the big disk.

$P_2(n)$ : Apply  $P_2(n - 1)$  to move the top  $n - 1$  disks two poles forward to land on the third pole. Then move the remaining big disk to the second pole. Then apply  $P_1(n - 1)$  to move the stack of  $n - 1$  disks one pole forward to land on the first pole. Now move the big disk 1 pole forward again to land on the third pole. Finally, apply  $P_2(n - 1)$  again to move the stack of  $n - 1$  disks two poles forward to land on the big disk.

(b) [5 pts] Let  $S_n$  be the number of moves needed to solve the  $n$ -disk problem. Express  $S_n$  with a recurrence equation and sufficient base cases.

**Solution.** For the first procedure, we have

$$\begin{aligned} S_1 &= 1, \\ S_n &= 2S_{n-1} + 1 + 2S_{n-1} = 4S_{n-1} + 1 \quad \text{for } n > 1. \end{aligned} \tag{1}$$

For the second procedure, in addition to the number,  $S_n$ , of steps to move a stack of  $n$  disks one pole forward a stack of  $n$ , let  $T_n$  be the number of steps to move a stack of  $n$  disks two poles forward. From the definitions of procedures  $P_1$  and  $P_2$  we have

$$\begin{aligned} S_1 &= 1, \\ T_1 &= 2, \\ S_n &= T_{n-1} + 1 + T_{n-1} \quad \text{for } n > 1, \end{aligned} \tag{2}$$

$$T_n = T_{n-1} + 1 + S_{n-1} + 1 + T_{n-1} \quad \text{for } n > 1. \tag{3}$$

From these equations we first calculate that  $T_2 = 7$ . Then, using (2) to substitute for  $S_{n-1}$  in (3), we conclude that for  $n > 2$ ,

$$T_n = 2T_{n-1} + 2 + (2T_{n-2} + 1) = 2T_{n-1} + 2T_{n-2} + 3. \tag{4}$$

(c) [5 pts] Find a closed-form expression for  $S_n$  by solving the recurrence.

**Solution.** For recurrence (1), Plug & Chug works nicely:

$$\begin{aligned} S_n &= 4S_{n-1} + 1 \\ &= 4(4S_{n-2} + 1) + 1 \\ &= 4^2S_{n-2} + 4 + 1 \\ &= 4^2(4S_{n-3} + 1) + 4 + 1 \\ &= 4^3S_{n-3} + 4^2 + 4 + 1 = \dots \\ &= 4^{n-1}S_1 + 4^{n-2} + \dots + 4^2 + 4 + 1 \\ &= 4^{n-1} + (4^{n-1} - 1)/3 \\ &= \frac{4^n - 1}{3} \end{aligned}$$

For recurrence (4), we apply the general approach for inhomogeneous linear recurrences: the characteristic polynomial is  $x^2 - 2x - 2$  with roots  $1 \pm \sqrt{3}$ , so the general solution to the homogenous part of (4) is

$$A(1 + \sqrt{3})^n + B(1 - \sqrt{3})^n.$$

Since the inhomogeneous term of (4) is constant, we guess that a particular solution will be some constant,  $c$ . This requires that  $c = 2c + 2c + 3$ , namely,  $c = -1$ . So the most general solution to (4) is of the form

$$A(1 + \sqrt{3})^n + B(1 - \sqrt{3})^n - 1. \tag{5}$$

Now we use the boundary conditions  $T_1 = 2, T_2 = 7$  and (5) to obtain linear equations in  $A$  and  $B$ :

$$\begin{aligned} A(1 + \sqrt{3}) + B(1 - \sqrt{3}) - 1 &= 2 \\ A(1 + \sqrt{3})^2 + B(1 - \sqrt{3})^2 - 1 &= 7. \end{aligned}$$

Solving these equations, we find  $A = (3 + 2\sqrt{3})/6$  and  $B = (3 - 2\sqrt{3})/6$ , so

$$T_n = \frac{3 + 2\sqrt{3}}{6}(1 + \sqrt{3})^n + \frac{3 - 2\sqrt{3}}{6}(1 - \sqrt{3})^n - 1.$$

Now using (2), we conclude that

$$S_n = \frac{3 + 2\sqrt{3}}{3}(1 + \sqrt{3})^{n-1} + \frac{3 - 2\sqrt{3}}{3}(1 - \sqrt{3})^{n-1} - 1.$$

In particular, we conclude that  $S_n = \Theta((1 + \sqrt{3})^n) = o(4^n)$ , so the second procedure for moving a stack of  $n$  disks is vastly more efficient than the first one.

**Problem 2. [30 points]** Find  $\Theta$  bounds for the following divide-and-conquer recurrences, which give expressions for  $T(n)$  for all integers  $n \geq 2$ . Assume that  $T(0) = T(1) = 1$  in all cases. Show your work.

- (a) [6 pts]  $T(n) = 3T(\lfloor n/9 \rfloor + 1) + \sqrt{n}$
- (b) [6 pts]  $T(n) = 7T(\lfloor n/2 \rfloor) + n^2$
- (c) [6 pts]  $T(n) = 36T(\lfloor n/6 + \sqrt{n} \rfloor) + n^2$
- (d) [6 pts]  $T(n) = T(\lfloor n/3 \rfloor) + T(\lfloor n/12 \rfloor) + 15n$
- (e) [6 pts]  $T(n) = 5T(\lfloor n/8 \rfloor) + n$

**Solution.**

(a)  $a_1 = 3, b_1 = 1/9, h_1(n) = \lfloor n/9 \rfloor + 1 - n/9, p = 1/2,$

$$T(n) = \Theta \left( n^{1/2} \left( 1 + \int_1^n \frac{u^{1/2}}{u^{3/2}} du \right) \right) = \Theta \left( n^{1/2} \left( 1 + \int_1^n \frac{1}{u} du \right) \right) = \Theta(\sqrt{n} \log n)$$

(b)  $a_1 = 7, b_1 = 1/2, h_1(n) = \lfloor n/2 \rfloor - n/2, p = \log_2 7,$

$$T(n) = \Theta \left( n^2 \left( 1 + \int_1^n \frac{u^2}{u^{\log_2 7 + 1}} du \right) \right) = \Theta \left( n^2 \left( 1 + \int_1^n \frac{1}{u^{\log_2 7 - 1}} du \right) \right) = \Theta(n^{\log_2 7})$$

(c)  $a_1 = 36, b_1 = 1/6, h_1(n) = \lfloor n/6 + \sqrt{n} \rfloor - n/6, p = 2,$

$$T(n) = \Theta \left( n^2 \left( 1 + \int_1^n \frac{u^2}{u^3} du \right) \right) = \Theta \left( n^2 \left( 1 + \int_1^n \frac{1}{u} du \right) \right) = \Theta(n^2 \log n)$$

(d)  $a_1 = 1, a_2 = 1, b_1 = 1/3, b_2 = 1/12, h_1(n) = \lfloor n/3 \rfloor - n/3, h_2(n) = \lfloor n/12 \rfloor - n/12,$   
 $g(n) = n, p < 1,$

$$T(n) = \Theta \left( n^p \left( 1 + \int_1^n \frac{15u}{u^{p+1}} du \right) \right) = \Theta \left( n^p \left( 1 + \int_1^n 15u^{-p} du \right) \right) = \Theta(n)$$

(e)  $a_1 = 5, b_1 = 1/8, h_1(n) = \lfloor n/8 \rfloor - n/8, p < 1,$

$$T(n) = \Theta \left( n^p \left( 1 + \int_1^n \frac{u}{u^{p+1}} du \right) \right) = \Theta \left( n^p \left( 1 + \int_1^n u^{-p} du \right) \right) = \Theta(n)$$

**Problem 3. [15 points]** Define the sequence of numbers  $A_i$  by

$$A_0 = 2$$

$$A_{n+1} = A_n/2 + 1/A_n \text{ (for } n \geq 1\text{)}$$

Prove that  $A_n \leq \sqrt{2} + 1/2^n$  for all  $n \geq 0$ .

**Solution.**

*Proof.* The proof is by induction on  $n$ . Let  $P(n)$  be the proposition that  $A_n \leq \sqrt{2} + 1/2^n$ .

**Base case:**  $A_0 = 2 \leq \sqrt{2} + 1/2^0$  is true.

**Inductive step:** Let  $n \geq 0$  and assume the inductive hypothesis  $A_n \leq \sqrt{2} + 1/2^n$ . We need the following lemma.

**Lemma.** For real numbers  $x > 0$ ,  $x/2 + 1/x \geq \sqrt{2}$ .

*Proof.* For real numbers  $x > 0$ ,

$$\begin{aligned} x/2 + 1/x &\geq \sqrt{2} \\ \Leftrightarrow x^2 + 2 &\geq 2\sqrt{2} \cdot x \\ \Leftrightarrow x^2 - 2\sqrt{2} \cdot x + 2 &\geq 0 \\ \Leftrightarrow (x - \sqrt{2})^2 &\geq 0, \end{aligned}$$

which is true. □

By using induction it is straightforward to prove that  $A_n > 0$  for  $n \geq 0$  (base case:  $A_0 = 2 > 0$ ; inductive step: if  $A_n > 0$ , then  $A_{n+1} = A_n/2 + 1/A_n > 0$ ). By the lemma,  $A_n \geq \sqrt{2}$  for  $n \geq 0$ . Together with the inductive hypothesis this can be used in the following derivation:

$$\begin{aligned} A_{n+1} &= A_n/2 + 1/A_n \\ &\leq (\sqrt{2} + 1/2^n)/2 + 1/\sqrt{2} \\ &= \sqrt{2} + 1/2^{n+1}. \end{aligned}$$

This completes the proof. □

**Problem 4. [30 points]** Find closed-form solutions to the following linear recurrences.

(a) [15 pts]  $x_n = -4x_{n-1} + 3x_{n-2} + 18x_{n-3}$  ( $x_0 = 2, x_1 = 2, x_2 = 3$ )

Hint: -3 is a root.

**Solution.** The characteristic equation is  $r^3 + 4r^2 - 3r - 18 = 0$ . Solving a cubic equation can be messy process, but in this case the roots are easy to find:

$$\begin{aligned} r_1 &= -3 \\ r_2 &= -3 \\ r_3 &= 2 \end{aligned}$$

Therefore a general form for a solution is

$$x_n = A(-3)^n + Bn(-3)^n + C2^n.$$

Substituting the initial conditions into this general form gives a system of linear equations.

$$\begin{aligned} 2 &= A + C \\ 2 &= -3A - 3B + 2C \\ 3 &= 9A + 18B + 4C \end{aligned}$$

The solution to this linear system is  $A = 17/25$ ,  $B = -7/15$ , and  $C = 33/25$ . The complete solution to the recurrence is therefore

$$x_n = \frac{17}{25}(-3)^n - \frac{7}{15}n(-3)^n + \frac{33}{25}2^n.$$

(b) [15 pts]  $x_n = 3x_{n-1} - 2x_{n-2} + n$  ( $x_0 = 0, x_1 = 1$ )

**Solution.** First, we find the general solution to the homogenous recurrence. The characteristic equation is  $r^2 - 3r + 2 = 0$ . The roots of this equation are  $r_1 = 1$  and  $r_2 = 2$ . Therefore, the general solution to the homogenous recurrence is

$$x_n = A + B2^n.$$

Now we must find a particular solution to the recurrence, ignoring the boundary conditions. Since the inhomogenous term is linear, we guess there is a linear solution, that is, a solution of the form  $an + b$ . If the solution is of this form, we must have

$$an + b = 3(a(n-1) + b) - 2(a(n-2) + b) + n,$$

which implies the absurd conclusion that

$$n = -a.$$

So we make another guess, this time that there is a quadratic solution of the form  $an^2 + bn + c$ . If the solution is of this form, we must have

$$an^2 + bn + c = 3(a(n-1)^2 + b(n-1) + c) - 2(a(n-2)^2 + b(n-2) + c) + n,$$

which simplifies to

$$(2a + 1)n + (b - 5a) = 0.$$

This last equation is satisfied only if the coefficient of  $n$  and the constant term are both zero, which implies  $a = -1/2$  and  $b = -5/2$ . Apparently, any value of  $c$  gives a valid particular solution. For simplicity, we choose  $c = 0$  and obtain the particular solution:

$$x_n = -\frac{1}{2}n^2 - \frac{5}{2}n.$$

The complete solution to the recurrence is the homogenous solution plus the particular solution:

$$x_n = A + B2^n - \frac{1}{2}n^2 - \frac{5}{2}n$$

Substituting the initial conditions gives a system of linear equations:

$$\begin{aligned} 0 &= A + B \\ 1 &= A + 2B - 3 \end{aligned}$$

The solution to this linear system is  $A = -4$  and  $B = 4$ . Therefore, the complete solution to the recurrence is

$$x_n = 2^{n+2} - \frac{1}{2}n^2 - \frac{5}{2}n - 4.$$

**Problem 5. [10 points]** The following recurrence was invented by a late Italian renaissance scholar named Giardi Invertabinacci, who earned a certain repute in his age for mocking profound ideas, poetry, sculpture, and even architecture by constructing upside-down or backward replicas of the original. His scaled model of the famed Tower of Pisa still stands—upside down, but not leaning—outside Palermo. His one venture into mathematics was this

recurrence relation, recorded in a letter to a contemporary mathematician and reproduced below in modern notation.<sup>1</sup>

$$\begin{aligned}x_1 &= 1 \\x_2 &= 1 \\x_{n+1} &= \frac{x_n x_{n-1}}{x_n + x_{n-1}}\end{aligned}$$

(a) [5 pts] Guess the solution to the recurrence. Your solution may be expressed in terms of Fibonacci numbers.

**Solution.** We compute the first few terms to form the basis for a guess at the general solution.

$$\begin{aligned}x_1 &= 1 \\x_2 &= 1 \\x_3 &= \frac{x_2 x_1}{x_2 + x_1} = \frac{1 \cdot 1}{1 + 1} = \frac{1}{2} \\x_4 &= \frac{x_3 x_2}{x_3 + x_2} = \frac{\frac{1}{2} \cdot 1}{\frac{1}{2} + 1} = \frac{1}{3} \\x_5 &= \frac{x_4 x_3}{x_4 + x_3} = \frac{\frac{1}{3} \cdot \frac{1}{2}}{\frac{1}{3} + \frac{1}{2}} = \frac{1}{5} \\x_6 &= \frac{x_5 x_4}{x_5 + x_4} = \frac{\frac{1}{5} \cdot \frac{1}{3}}{\frac{1}{5} + \frac{1}{3}} = \frac{1}{8}\end{aligned}$$

**Guess.** The solution to the recurrence is  $x_n = 1/F_n$ .

(b) [5 pts] Use strong induction to prove your guess correct.

**Solution.**

*Proof.* The proof is by strong induction. Let  $P(n)$  be the proposition that  $x_n = 1/F_n$ . In the base cases,  $P(1)$  is true because  $x_1 = 1 = 1/F_1$ , and  $P(2)$  is true because  $x_2 = 1 = 1/F_2$ . In the inductive step, for  $n \geq 2$  assume  $P(1), \dots, P(n)$  to prove  $P(n+1)$ . We reason as

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<sup>1</sup>Yeah, yeah, we made all this up. But good story, eh?

follows:

$$\begin{aligned}x_{n+1} &= \frac{x_n x_{n-1}}{x_n + x_{n-1}} && \text{(def. of } x_{n+1}\text{)} \\ &= \frac{(1/F_n)(1/F_{n-1})}{1/F_n + 1/F_{n-1}} && \text{(ind. hyp.)} \\ &= \frac{1}{F_{n-1} + F_n} && \text{(simplification)} \\ &= \frac{1}{F_{n+1}} && \text{(Fibonacci recurrence)}\end{aligned}$$

□