Problem Set 6

Due: Tuesday, October 14

Problem 1. [15 points] For each of the following, either prove that it is an equivalence relation and state its equivalence classes, or give an example of why it is not an equivalence relation.

(a) [5 pts] $R = \{(x, y) \in W \times W$ such that the words x and y start with the same letter$\}$, where W is the set of all words in the 2001 edition of the Oxford English dictionary.

(b) [5 pts] $S = \{(x, y) \in W \times W$ such that the words x and y have at least one letter in common$\}$.

(c) [5 pts] $R := \{(x, y) \in \mathbb{R} \times \mathbb{R}$ such that $\exists n \in \mathbb{Z}, y = 2^n x\}$.

Problem 2. [20 points] In this problem we study partial orders (posets). Recall that a partial order $\preceq$ on a set X is reflexive ($x \preceq x$), anti-symmetric ($x \preceq y \wedge y \preceq x \rightarrow x = y$), and transitive ($x \preceq y \wedge y \preceq z \rightarrow x \preceq z$). Note that it may be the case that neither $x \preceq y$ nor $y \preceq x$. A chain is a list of *distinct* elements $x_1, \ldots, x_i$ in X for which $x_1 \preceq x_2 \preceq \cdots \preceq x_i$. An antichain is a subset S of X such that for all distinct $x, y \in S$, neither $x \preceq y$ nor $y \preceq x$.

The aim of this problem is to show that any sequence of $(n-1)(m-1)+1$ integers either contains a non-decreasing subsequence of length n or a decreasing subsequence of length m. Note that the given sequence may be out of order, so, for instance, it may have the form $1, 5, 3, 2, 4$ if $n = m = 3$. In this case the longest non-decreasing and longest decreasing subsequences have length 3 (for instance, consider $1, 2, 4$ and $5, 3, 2$).

(a) [5 pts] Label the given sequence of $(n-1)(m-1)+1$ integers $a_1, a_2, \ldots, a_{(n-1)(m-1)+1}$. Show the following relation $\preceq$ on $\{1, 2, 3, \ldots, (n-1)(m-1)+1\}$ is a poset: $i \preceq j$ if and only if $i \leq j$ and $a_i \leq a_j$ (as integers).

For the next part, we will need to use Dilworth's theorem: Dilworth's theorem states that if $(X, \preceq)$ is any poset whose longest chain has length n, then X can be partitioned into at most n disjoint antichains.

(b) [10 pts] Show that in any sequence of $(n-1)(m-1)+1$ integers, either there is a non-decreasing subsequence of length n or a decreasing subsequence of length m.

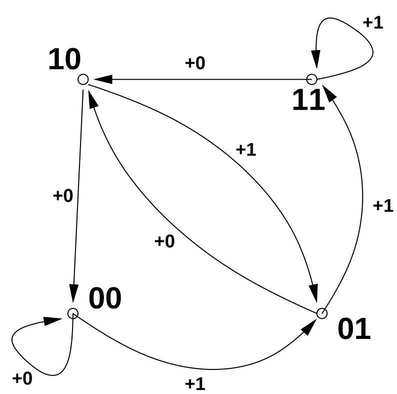
(c) [5 pts] Construct a sequence of $(n-1)(m-1)$ integers, for arbitrary n and m, that has no non-decreasing subsequence of length n and no decreasing subsequence of length m. Thus in general, the result you obtained in the previous part is best-possible.

Problem 3. [**15 points**]

A 3-bit string is a string made up of 3 characters, each a 0 or a 1. Suppose you'd like to write out, in one string, all eight of the 3-bit strings in any convenient order. For example, if you wrote out the 3-bit strings in the usual order starting with 000 001 010..., you could concatenate them together to get a length $3 \cdot 8 = 24$ string that started 000001010....

But you can get a shorter string containing all eight 3-bit strings by starting with 00010.... Now 000 is present as bits 1 through 3, 001 is present as bits 2 through 4, 010 is present as bits 3 through 5, ...

(a) [3 pts] Take a few moments to see how short a string you can make that contains every 3-bit string as 3 consecutive bits somewhere in it. Explain why 10 bits is the absolute minimum length for such a string.



(b) [3 pts] Imagine that the labels on the vertices of the directed graph below represent the last two digits in a string you build by adding one bit at a time. Explain why the graph completely describes how the last two digits of your string can change throughout this process.

(c) [3 pts] Find a directed path in this graph starting at some vertex, v, that traverses every edge exactly once. Note that vertices will have to be used more than once and the path wil have to end in v.

(d) [3 pts] Explain how such a path provides a shortest possible solution to the original problem.

(e) [3 pts] What about k-bit substrings, $k = 4, 5, \ldots$? Can you define the appropriate generalization of the useful graph above? (They're called de Bruijn graphs.) If you do it sucessfully, you should be able to see that the in-degree (as well as the out-degree) of every vertex is 2.

Problem 4. [**30 points**] Louis Reasoner figures that, wonderful as the Beneš network may be, the butterfly network has a few advantages, namely: fewer switches, smaller diameter, and an easy way to route packets through it. So Louis designs an N-input/output network he modestly calls a *Reasoner-net* with the aim of combining the best features of both the butterfly and Beneš nets:

> The ith input switch in a Reasoner-net connects to two switches, a_i and b_i, and likewise, the jth output switch has two switches, y_j and z_j, connected to it. Then the Reasoner-net has an N-input Beneš network connected using the a_i switches as input switches and the y_j switches as its output switches. The Reasoner-net also has an N-input butterfly net connected using the b_i switches as inputs and the z_j switches as outputs.

In the Reasoner-net the minimum latency routing does not have minimum congestion. The *latency for min-congestion* (LMC) of a net is the best bound on latency achievable using routings that minimize congestion. Likewise, the *congestion for min-latency* (CML) is the best bound on congestion achievable using routings that minimize latency.

Fill in the following chart for the Reasoner-net and briefly explain your answers.

diameter	**switch size(s)**	**# switches**	**congestion**	**LMC**	**CML**

Problem 5. [20 points] Let B_n denote the butterfly network with $N = 2^n$ inputs and N outputs, as defined in Lecture Notes 10. We will show that the congestion of B_n is exactly $\sqrt{N}$ when n is even.

Hints:

- For the butterfly network, there is a unique path from each input to each output, so the congestion is the maximum number of messages passing through a vertex for any matching of inputs to outputs.
- If v is a vertex at level i of the butterfly network, there is a path from exactly 2^i input vertices to v and a path from v to exactly 2^{n-i} output vertices.
- At which level of the butterfly network must the congestion be worst? What is the congestion at the node whose binary representation is all 0s at that level of the network?

(a) [10 pts] Show that the congestion of B_n is at most $\sqrt{N}$ when n is even.

(b) [10 pts] Show that the congestion achieves $\sqrt{N}$ somewhere in the network and conclude that the congestion of B_n is exactly $\sqrt{N}$ when n is even.