

Problem Set 4

Due: Monday, September 29

Problem 1. [15 points] Let $G = (V, E)$ be a graph. A *matching* in G is a set $M \subset E$ such that no two edges in M are incident on a common vertex.

Let M_1, M_2 be two matchings of G . Consider the new graph $G' = (V, M_1 \cup M_2)$ (i.e. on the same vertex set, whose edges consist of all the edges that appear in either M_1 or M_2). Show that G' is bipartite.

We will need this result in one of the coming lectures.

Problem 2. [15 points] Let $G = (V, E)$ be a graph. Recall that the *degree* of a vertex $v \in V$, denoted d_v , is the number of vertices w such that there is an edge between v and w .

(a) [5 pts] Prove that

$$2|E| = \sum_{v \in V} d_v.$$

(b) [5 pts] At a 6.042 ice-cream study session (where ice-cream flows by the way, really, you should go ... and yeah, it helps you study too) 111 students showed up. During the session, some students shook hands with each other (everybody being happy and content with the ice-cream and all). Turns out that the University of Chicago did another spectacular study here, and counted that each student shook hands with exactly 17 other students. Can you debunk this too?

(c) [5 pts] And on a more dull note, how many edges does K_n , the complete graph on n vertices, have?

Problem 3. [30 points] Let n be a positive integer. Consider the graph G whose vertices are the elements of $\{1, 2, \dots, 2n\}$, and whose edges are given by the following rule: there is an edge between vertex i and j iff $((i - j \equiv 1 \pmod{2n}) \vee (i - j \equiv -1 \pmod{2n}) \vee (i - j \equiv n \pmod{2n}))$.

(a) [5 pts] For each $k \in \{1, 2, \dots, 2n\}$, find the distance between vertex 1 and vertex k .

(b) [5 pts] Prove that this graph is not 4-edge-connected: that is, you can remove 3 edges and disconnect the graph.

(c) [5 pts] Prove that this graph is 3-edge-connected: that is, if you remove two edges from the graph, the graph remains connected.

(d) [5 pts] Describe the induced subgraph on the odd numbered vertices $\{1, 3, \dots, 2n-1\}$.

(e) [5 pts] Describe the induced subgraph on the vertices $\{1, 2, \dots, n\}$.

(f) [5 pts] What is the chromatic number of G ? (It may depend on n).

Problem 4. [20 points] A planar graph is one which can be drawn in the plane without any edges crossing (i.e. without the lines or arcs representing them intersecting except at common endpoints). Any planar graph with n vertices and m edges satisfies $m \leq 3n - 6$. Show that

(a) [5 pts] any planar graph has a node of degree at most 5.

(b) [15 pts] Using induction, prove that any planar graph can be colored with six colors.

Problem 5. [20 points] In a set of stable marriages between an equal number of boys and girls, call a person *lucky* if their spouse appears in the top half of their preference list.

Claim. *The Mating Algorithm produces a set of marriages with at least one lucky person.*

To prove the Claim, for each girl, G , define a “rejection count” derived variable, $r(G)$, to be the number of boys she has rejected. Similarly, for each boy, B , define a “rejected count” variable, $r(B)$, to be the number of times he has been rejected by girls.

(a) [6 pts] Define the predicate $L(B)$ meaning “ B is a lucky boy,” in terms of the final value of $r(B)$.

(b) [6 pts] Suppose that on the final day, the value of $r(G)$, averaged over all the girls, is at *least* half the number of boys. Explain why there must be a lucky girl.

(c) [8 pts] The rejection counts in the Mating Algorithm satisfy an obvious invariant. Use this invariant and the previous problem parts to prove the Claim.