Problem Set 4 Solutions

Due: Monday, September 29

Problem 1. [15 points] Let $G = (V, E)$ be a graph. A *matching* in G is a set $M \subset E$ such that no two edges in M are incident on a common vertex.

Let M_1, M_2 be two matchings of G. Consider the new graph $G' = (V, M_1 \cup M_2)$ (i.e. on the same vertex set, whose edges consist of all the edges that appear in either M_1 or M_2). Show that G' is bipartite.

We will need this result in one of the coming lectures.

Solution. We will show that G' has no odd cycle. By the theorem proved in recitation (that a graph is bipartite iff it has no odd cycles) we will be done.

First, consider any vertices connected by an edge that is in both M_1 and M_2. Since these are matchings, there is no other edges connected to these vertices in either M_1 or M_2 and thus G'. These vertices form connected components of size two which have no odd length cycles. For the rest of the proof, we can assume that every edge in G' came from exactly one of M_1 or M_2.

Take a sequence of edges $(e_1, e_2, \ldots, e_k)$ in G' that form a cycle. Let us show that k must be even. We know that $e_1 \in M_1 \cup M_2$. Assume $e_1 \in M_1$ (otherwise $e_1 \in M_2$ and the argument is identical replacing M_1 with M_2). Now since e_1 and e_2 are incident on a common vertex, and M_1 is a matching, e_2 cannot be in M_1. So $e_2 \in M_2$. Similarly $e_3 \in M_1$, $e_4 \in M_2$ etc. By induction we can prove that for all $i \leq k$, $e_i \in M_1$ iff i is odd.

Now if k had been odd, we would have $e_k \in M_1$. But e_k and e_1 are adjacent edges in the cycle, and hence incident on a common vertex, we have a contradiction. Thus k must be even. ■

Problem 2. [15 points] Let $G = (V, E)$ be a graph. Recall that the *degree* of a vertex $v \in V$, denoted d_v, is the number of vertices w such that there is an edge between v and w.

(a) [5 pts] Prove that

$$2|E| = \sum_{v \in V} d_v.$$

Solution. Let $S = \{(e, v) \in E \times V : e \text{ is incident on } v\}$.

Count the elements in S as follows

$$|S| = \sum_{e \in E} |\{v : (e, v) \in S\}| = 2|E|$$

and also as

$$|S| = \sum_{v \in V} |\{e : (e, v) \in S\}| = \sum_{v \in V} d_v.$$

The result follows.

One can also prove it by induction on $|E|$. You should try to. ■

(b) [5 pts] At a 6.042 ice-cream study session (where ice-cream flows by the way, really, you should go ... and yeah, it helps you study too) 111 students showed up. During the session, some students shook hands with each other (everybody being happy and content with the ice-cream and all). Turns out that the University of Chicago did another spectacular study here, and counted that each student shook hands with exactly 17 other students. Can you debunk this too?

Solution. Assume that the study is accurate. Define a graph $G = (V, E)$ with students as vertices and put an edge between 2 students if they shook hands. By the previous problem, we should have $2|E| = \sum_v d_v = 111 \cdot 17$. But the LHS is even and the RHS is odd, a contradiction. ■

(c) [5 pts] And on a more dull note, how many edges does K_n, the complete graph on n vertices, have?

Solution. Apply the first part of the problem. Notice that each vertex is joined to $n - 1$ others. $2|E| = \sum_v d_v = n(n-1)$. So $|E| = n(n-1)/2$. ■

Problem 3. [30 points] Let n be a positive integer. Consider the graph G whose vertices are the elements of $\{1, 2, \ldots, 2n\}$, and whose edges are given by the following rule: there is an edge between vertex i and j iff ($(i - j \equiv 1 \mod 2n) \vee (i - j \equiv -1 \mod 2n) \vee (i - j \equiv n \mod 2n)$).

(a) [5 pts] For each $k \in \{1, 2, \ldots, 2n\}$, find the distance between vertex 1 and vertex k.

Solution. If $k \leq \lfloor n/2 \rfloor + 1$, the distance is $k - 1$. If $\lfloor n/2 \rfloor + 1 < k \leq n + 1$, the distance is $n - k + 2$. If $n + 1 < k \leq \lceil 3n/2 \rceil$, the distance is $k - n$. If $\lceil 3n/2 \rceil < k \leq 2n$, the distance is $2n - k + 1$. ■

(b) [5 pts] Prove that this graph is not 4-edge-connected: that is, you can remove 3 or fewer edges and disconnect the graph.

Solution. Consider vertex 1. If $n = 1$, remove its single edge to disconnect the graph, otherwise remove the 3 edges adjacent to vertex 1. ■

(c) [5 pts] Prove that this graph is 3-edge-connected for all $n > 1$: that is, if you remove two or fewer edges from the graph, the graph remains connected.

Solution. Suppose you could remove 2 edges and disconnect the graph. Now if only 1 edge is removed from the big cycle, the big cycle remains connected (and hence the graph too). Thus it must be that both the edges were removed from the big cycle. This breaks the big cycle into 2 components. However, there is still a diameter edge crossing from one component to the other, a contradiction. ■

(d) [5 pts] Describe the induced subgraph on the odd numbered vertices $\{1, 3, \ldots, 2n-1\}$.

Solution. If n is odd, then it is n disconnected vertices. If n is even it is a perfect matching on n vertices. ■

(e) [5 pts] Describe the induced subgraph on the vertices $\{1, 2, \ldots, n\}$.

Solution. The induced subgraph is a chain of length $n-1$ connecting every vertex $i \in 1, ..., n-1$ to $i+1$. There are no other edges. ■

(f) [5 pts] What is the chromatic number of G? (It may depend on n).

Solution. If n is odd, it is 2 colorable (color vertex i by color $rem(i, 2)$). If $n = 2$ it is 4 colorable. If n is even and ≥ 4, it is 3 colorable (color vertex $1, \ldots, n$ with colors $1, 2, 1, 2, \ldots, 1, 2$ and color vertices $n+1, \ldots, 2n-2$ with colors $3, 1, 3, 1, \ldots, 3, 1$ and color vertices $2n-1$ and $2n$ with colors $2, 3$. It can be checked that this is a coloring). ■

Problem 4. [20 points] A planar graph is one which can be drawn in the plane without any edges crossing (i.e. without the lines or arcs representing them intersecting except at common endpoints). Any planar graph with n vertices and m edges satisfies $m \leq 3n - 6$. Show that

(a) [5 pts] any planar graph has a node of degree at most 5.

Solution. Suppose that every vertex has degree at least 6. Then

$$\begin{aligned} 2m &= \sum_{v \in V} \deg(v) \geq \sum_{v \in V} 6 = 6n \\ m &\geq 3n \end{aligned}$$

contradicting our assertion that $m \leq 3n - 6$. ■

(b) [15 pts]Using induction, prove that any planar graph can be colored with six colors.

Solution. The proof is by induction on the number of vertices n.

$P(n) =$ "A planar graph of n vertices can be colored with at most 6 colors".

Base case: $P(1)$ is true because a single vertex can be colored with 1 color.

Inductive step: Assume $P(n)$ is true in order to show that $P(n+1)$ is true.

Let G be a planar graph with $n+1$ vertices and remove a vertex v of degree 5 (or less). The remaining n-vertex graph can be colored with 6 colors by the inductive hypothesis. Re-attach v. v is adjacent to at most 5 vertices, occupying at most 5 out of the six colors. We can use the remaining color to color v.

We have shown that $P(n) \to P(n+1)$, so the proof is complete.

■

Problem 5. [20 points] In a set of stable marriages between an equal number of boys and girls, call a person *lucky* if their spouse appears in the top half of their preference list.

Claim. *The Mating Algorithm produces a set of marriages with at least one lucky person.*

To prove the Claim, for each girl, G, define a "rejection count" derived variable, $r(G)$, to be the number of boys she has rejected. Similarly, for each boy, B, define a "rejected count" variable, $r(B)$, to be the number of times he has been rejected by girls.

(a) [6 pts] Define the predicate $L(B)$ meaning "B is a lucky boy," in terms of the final value of $r(B)$.

Solution. Since a boy works down his list of favorite girls, he will be in the top half of his list if he has been rejected by fewer than half the girls:

$$L(B) = \text{ "the value of } r(B) \text{ on the final day is less than half the number of girls".}$$

■

(b) [6 pts] Suppose that on the final day, the value of $r(G)$, averaged over all the girls, is at *least* half the number of boys. Explain why there must be a lucky girl.

Solution. Suppose there was no lucky girl. Then the value of $r(G)$ for each girl would be less than $n/2$, since if any girl rejected at least $n/2$ boys, she would be married to a boy in the top half of her list, and therefore she would be lucky. But if the value of $r(G)$ is less than $n/2$ for each girl, then the average of $r(G)$ would be less than $n/2$, which contradicts the fact that the average of $r(G)$ is at least $n/2$. ■

(c) [8 pts] The rejection counts in the Mating Algorithm satisfy an obvious invariant. Use this invariant and the previous problem parts to prove the Claim.

Solution. At any stage of the Mating Algorithm, the total number of rejections by girls must equal the total number of times boys get rejected:

$$\sum_G r(G) = \sum_B r(B).$$

Now if no boy is lucky, then by part (**??**), the final value (on the wedding day) of $\sum_B r(B)$ is at least $n(n/2)$ where n is the number of boys. So the invariant implies that on the wedding day $\sum_G r(G) > n(n/2)$. So the average value of $r(G)$ on the wedding day is at least $n/2$, and part (**??**) implies there must be a lucky girl. ■