6.042/18.062J Mathematics for Computer Science September 9, 2008
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Problem Set 2

Due: Monday, September 15

Problem 1. [**10 points**] Use induction to prove that

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{n}\right)=\frac{1}{n}$$

for all $n \geq 2$.

Problem 2. [**25 points**] The following problem is fairly tough until you hear a certain one-word clue. The solution is elegant but is slightly tricky, so don't hesitate to ask for hints!

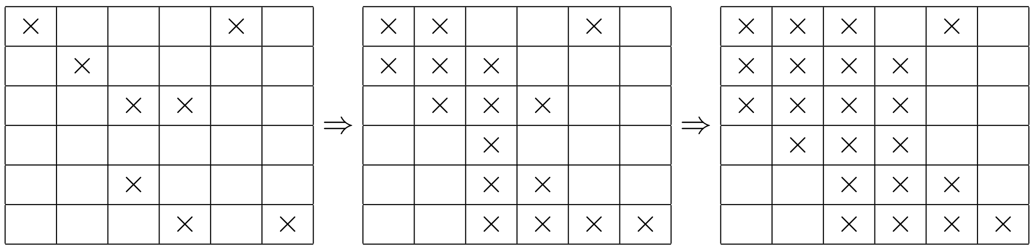
During 6.042, the students are sitting in an $n \times n$ grid. A sudden outbreak of beaver flu (a rare variant of bird flu that lasts forever; symptoms include yearning for problem sets and craving for ice cream study sessions) causes some students to get infected. Here is an example where $n = 6$ and infected students are marked $\times$.

$\times$				$\times$	
	$\times$				
		$\times$	$\times$		
		$\times$			
			$\times$		$\times$

Now the infection begins to spread every minute (in discrete time-steps). Two students are considered *adjacent* if they share an edge (i.e., front, back left or right, but NOT diagonal); thus, each student is adjacent to 2, 3 or 4 others. A student is infected in the next time step if either

- the student was previously infected (since beaver flu lasts forever), or
- the student is adjacent to *at least two* already-infected students.

In the example, the infection spreads as shown below.



In this example, over the next few time-steps, all the students in class become infected.

Theorem. *If fewer than n students in class are initially infected, the whole class will never be completely infected.*

Prove this theorem.

Hint: When one wants to understand how a system such as the above "evolves" over time, it is usually a good strategy to (1) identify an appropriate property of the system at the initial stage, and (2) prove, by induction on the number of time-steps, that the property is preserved at every time-step. So look for a property (of the set of infected students) that remains invariant as time proceeds.

If you are stuck, ask your recitation instructor for the one-word clue and even more hints!

Problem 3. [**15 points**] Define the sequence of numbers N_i, by

$N_0 = 0$

$N_1 = 1$

$N_{n+1} = N_n^2 + N_n N_{n-1} + N_{n-1}^2.$

(a) [**5 points**] Prove that if $n \geq 1$, then N_n is odd.

(b) [**10 points**] Prove that if $3|n$, then $3|N_n$.

Problem 4. [**15 points**] Suppose we want to divide a class of n students into groups each containing either 4 or 5 students.

(a) [**5 points**] Let's try to use strong induction to prove that a class with $n \geq 8$ students can be divided into groups of 4 or 5.

Proof. The proof is by strong induction. Let $P(n)$ be the proposition that a class with n students can be divided into teams of 4 or 5.

Base case: We prove that $P(n)$ is true for $n = 8$, 9, or 10 by showing how to break classes of these sizes into groups of 4 or 5 students:

$$\begin{aligned} 8 &= 4+4 \\ 9 &= 4+5 \\ 10 &= 5+5 \end{aligned}$$

Inductive step: We must show that $P(8), \ldots, P(n)$ imply $P(n+1)$ for all $n \geq 10$. Thus, we assume that $P(8), \ldots, P(n)$ are all true and show how to divide up a class of $n+1$ students into groups of 4 or 5. We first form one group of 4 students. Then we can divide the remaining $n-3$ students into groups of 4 or 5 by the assumption $P(n-3)$. This proves $P(n+1)$, and so the claim holds by induction. □

This proof contains a critical logical error. Identify the first sentence in the proof that does not follow and explain what went wrong. (Pointing out that the *claim* is false is not sufficient; you must find the first logical error in the *proof*.)

(b) [**10 points**] Provide a correct strong induction proof that a class with $n \geq 12$ students can be divided into groups of 4 or 5.

Problem 5. [**20 points**] In the 15-puzzle, there are 15 lettered tiles and a blank square arranged in a 4×4 grid. Any lettered tile adjacent to the blank square can be slid into the blank. For example, a sequence of two moves is illustrated below:

A	B	C	D
E	F	G	H
I	J	K	L
M	O	N	

→

A	B	C	D
E	F	G	H
I	J	K	L
M	O		**N**

→

A	B	C	D
E	F	G	H
I	J		L
M	O	**K**	N

In the leftmost configuration shown above, the O and N tiles are out of order. Using only legal moves, is it possible to swap the N and the O, while leaving all the other tiles in their original position and the blank in the bottom right corner? In this problem, you will prove the following theorem stating that the answer is "no".

Theorem. *No sequence of moves transforms the board below on the left into the board below on the right.*

A	B	C	D
E	F	G	H
I	J	K	L
M	**O**	**N**	

A	B	C	D
E	F	G	H
I	J	K	L
M	**N**	**O**	

(a) [**2 points**] We define the "order" of the tiles in a board to be the sequence of tiles on the board reading from the top row to the bottom row and from left to right within a row. For example, in the right board depicted in the above theorem, the order of the tiles is A, B, C, D, E, etc.

Can a row move change the order of the tiles? Prove your answer.

(b) [2 points] How many pairs of tiles will have their relative order changed by a column move? More formally, for how many pairs of letters L_1 and L_2 will L_1 appear earlier in the order of the tiles than L_2 before the column move and later in the order after the column move? Prove your answer correct.

(c) [2 points] We define an *inversion* to be a pair of letters L_1 and L_2 for which L_1 precedes L_2 in the alphabet, but L_1 appears after L_2 in the order of the tiles. For example, consider the following configuration:

A	B	C	E
D	H	G	F
I	J	K	L
M	N	O	

There are exactly four inversions in the above configuration: E and D, H and G, H and F, and G and F.

What effect does a row move have on the parity of the number of inversions? Prove your answer.

(d) [4 points] What effect does a column move have on the parity of the number of inversions? Prove your answer.

(e) [8 points] The previous problem part implies that we must make an *odd* number of column moves in order to exchange just one pair of tiles (N and O, say). But this is problematic, because each column move also knocks the blank square up or down one row. So after an *odd* number of column moves, the blank can not possibly be back in the last row, where it belongs! Now we can bundle up all these observations and state an invariant, a property of the puzzle that never changes, no matter how you slide the tiles around.

Lemma. *In every configuration reachable from the position shown below, the parity of the number of inversions is different from the parity of the row containing the blank square.*

row 1	A	B	C	D
row 2	E	F	G	H
row 3	I	J	K	L
row 4	M	O	N	

Prove this lemma.

(f) [2 points] Prove the theorem that we originally set out to prove.

Problem 6. [15 points]

When exploring the basement of MIT, you find a mysterious room. The room has three desks. On one desk is a red bottle and a blue bottle. On another desk, there is a book. It is open to a page with the following instructions:

Here lies the secret to great fortune, or, if misused, to great peril:

1. *Place 5 copper coins and 3 silver coins in a red bottle. The coins will disappear and 14 gold coins will appear in their place.*
2. *Place 7 gold coins in a blue bottle. The coins will disappear and 3 copper coins and 1 silver coin will appear in their place.*

The third desk has several coins. You count them, and find that there are 100 copper coins, 50 silver coins, and no gold coins.

Let a represent the number of copper coins, b represent the number of silver coins, and c represent the number of gold coins.

(a) [**5 points**] Describe the situation as a state machine, including the transitions.

(b) [**5 points**] Can there ever be 56 copper coins and 24 silver coins? Prove your answer. (Hint: Look at the expression $a + b$.)

(c) [**5 points**] Is it possible to end up with 270 or more gold coins? Prove your answer. (Hint: Look at the expression $7(a + b) + 4c$.)