Problem Set 2 Solutions

Due: Monday, September 15

Problem 1. [**10 points**] Use induction to prove that

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{n}\right)=\frac{1}{n}$$

for all $n \geq 2$.

Solution.

Proof. The proof is by induction on n. Let $P(n)$ be the proposition that the equation above holds.

Base case: $P(2)$ is true because

$$\left(1-\frac{1}{2}\right)=\frac{1}{2}$$

Inductive step: Assume $P(n)$ is true. Then we can prove $P(n+1)$ is also true as follows:

$$\begin{aligned}\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n+1}\right) &= \frac{1}{n}\cdot\left(1-\frac{1}{n+1}\right)\\ &= \frac{1}{n+1}\end{aligned}$$

The first step uses the assumption $P(n)$ and the second is simplification.

Thus, $P(2)$ is true and $P(n)$ implies $P(n+1)$ for all $n \geq 2$. Therefore, $P(n)$ is true for all $n \geq 2$ by the principle of induction. □

Problem 2. [**25 points**] The following problem is fairly tough until you hear a certain one-word clue. The solution is elegant but is slightly tricky, so don't hesitate to ask for hints!

During 6.042, the students are sitting in an $n \times n$ grid. A sudden outbreak of beaver flu (a rare variant of bird flu that lasts forever; symptoms include yearning for problem sets and craving for ice cream study sessions) causes some students to get infected. Here is an example where $n = 6$ and infected students are marked $\times$.

×				×	
	×				
		×	×		
		×			
			×		×

Now the infection begins to spread every minute (in discrete time-steps). Two students are considered *adjacent* if they share an edge (i.e., front, back left or right, but NOT diagonal); thus, each student is adjacent to 2, 3 or 4 others. A student is infected in the next time step if either

- the student was previously infected (since beaver flu lasts forever), or
- the student is adjacent to *at least two* already-infected students.

In the example, the infection spreads as shown below.

×				×	
	×				
		×	×		
		×			
			×		×

$\Rightarrow$

×	×			×	
×	×	×			
	×	×	×		
		×			
		×	×		
		×	×	×	×

$\Rightarrow$

×	×	×		×	
×	×	×	×		
×	×	×	×		
	×	×	×		
		×	×	×	
		×	×	×	×

In this example, over the next few time-steps, all the students in class become infected.

Theorem. *If fewer than n students in class are initially infected, the whole class will never be completely infected.*

Prove this theorem.

Hint: When one wants to understand how a system such as the above "evolves" over time, it is usually a good strategy to (1) identify an appropriate property of the system at the initial stage, and (2) prove, by induction on the number of time-steps, that the property is preserved at every time-step. So look for a property (of the set of infected students) that remains invariant as time proceeds.

If you are stuck, ask your recitation instructor for the one-word clue and even more hints!

Solution.

Proof. Define the *perimeter* of an infected set of students to be the number of edges with infection on exactly one side. Let I denote the perimeter of the initially-infected set of students.

Now we use induction on the number of time steps to prove that the perimeter of the infected region never increases. Let $P(k)$ be the proposition that after k time steps, the perimeter of the infected region is at most I.

Base case: $P(0)$ is true by definition; the perimeter of the infected region is at most I after 0 time steps, because I is defined to be the perimeter of the initially-infected region.

Inductive step: Now we must show that $P(k)$ implies $P(k+1)$ for all $k \geq 0$. So assume that $P(k)$ is true, where $k \geq 0$; that is, the perimeter of the infected region is at most I after k steps. The perimeter can only change at step $k+1$ because some squares are newly infected. By the rules above, each newly-infected square is adjacent to at least two previously-infected squares. Thus, for each newly-infected square, at least two edges are removed from the perimenter of the infected region, and at most two edges are added to the perimeter. Therefore, the perimeter of the infected region can not increase and is at most I after $k+1$ steps as well. This proves that $P(k)$ implies $P(k+1)$ for all $k \geq 0$.

By the principle of induction, $P(k)$ is true for all $k \geq 0$.

If an $n \times n$ grid is completely infected, then the perimeter of the infected region is $4n$. Thus, the whole grid can become infected only if the perimeter is initially at least $4n$. Since each square has perimeter 4, at least n squares must be infected initially for the whole grid to be infected. □

The above proof shows that if initially k students are infected, then the perimeter of the infected region will never exceed $4k$. The largest number of students that can be contained within a region with perimeter $\leq 4k$ is equal to k^2, therefore, if k students in class are initially infected, then at most k^2 students will eventually be infected. This feels intuitively true after having done the previous proof. However, to give a formal proof requires some case analysis (try it!).

Problem 3. [15 points] Define the sequence of numbers N_i, by

$$N_0 = 0$$

$$N_1 = 1$$

$$N_{n+1} = N_n^2 + N_n N_{n-1} + N_{n-1}^2.$$

(a) [5 points] Prove that if $n \geq 1$, then N_n is odd.

Solution.

Proof. The proof is by strong induction on n. Let $P(n)$ be the proposition that N_n is odd.

Base case: $P(1)$ is true because $N_1 = 1$ is odd. $P(2)$ is true because $N_2 = N_1^2 + N_1 N_0 + N_0^2 = 1$ is odd.

Inductive step: Let $n \geq 2$ and suppose $P(k)$ is true for $1 \leq k \leq n$ in order to show that $P(n+1)$ is true. Notice that a product of two integers is an odd number.

So, the validity of $P(n)$ and $P(n-1)$ implies that N_n^2, $N_n N_{n-1}$, and N_{n-1}^2 are odd. Adding three odd integers together results again in an odd integer. So, the sum $N_n^2 + N_n N_{n-1} + N_{n-1}^2 = N_{n+1}$ is odd which proves the induction step.

This completes the proof. □

(b) **[10 points]** Prove that if $3|n$, then $3|N_n$.

Solution. An honest effort will show that we need to strengthen the induction hypothesis.

Proof. Proof by strong induction on n. Let $P(n)$ be the proposition that

$$[3|n \Rightarrow 3|N_n] \text{ AND } [(3 \not| n) \Rightarrow (\exists_{a\in\mathbb{Z}}\ N_n = 3a+1)].$$

Base case: $P(0)$ is true because $3|0$ and $N_0 = 0$ is divisible by 3. $P(1)$ is true because $3 \not| 1$ and $N_1 = 1$.

Inductive step: Let $n \geq 1$ and suppose $P(k)$ is true for $0 \leq k \leq n$ in order to show that $P(n+1)$ is true.

We use case analysis. There are three cases:

1. $\exists_{d\in\mathbb{Z}}\ n+1 = 3d$.
   In this case, 3 does not divide n and $n-1$ so by the inductive hypothesis, there exist integers a and b such that $N_n = 3a+1$ and $N_{n-1} = 3b+1$. Therefore, $N_n^2 = 3(3a^2+2a)+1$, $N_n N_{n-1} = 3(3ab+a+b)+1$, and $N_{n-1}^2 = 3(3b^2+2b)+1$, so their sum, which is equal to N_{n+1}, is divisible by 3.
2. $\exists_{d\in\mathbb{Z}}\ n+1 = 3d+1$.
   In this case, 3 does not divide $n-1$, but it does divide n. By the inductive hypothesis, there exist integers a and b such that $N_n = 3a$ and $N_{n-1} = 3b+1$. Therefore, $N_n^2 = 3(3a^2)$, $N_n N_{n-1} = 3(3ab+a)$, and $N_{n-1}^2 = 3(3b^2+2b)+1$, so their sum, which is equal to N_{n+1}, is equal to a multiple of 3 plus 1.
3. $\exists_{d\in\mathbb{Z}}\ n+1 = 3d+2$.
   In this case, 3 does not divide n, but it does divide $n-1$. We use a similar argument as in the previous case: By the inductive hypothesis, there exist integers a and b such that $N_n = 3a+1$ and $N_{n-1} = 3b$. Therefore, $N_n^2 = 3(3a^2+2a)+1$, $N_n N_{n-1} = 3(3ab+b)$, and $N_{n-1}^2 = 3(3b^2)$, so their sum, which is equal to N_{n+1}, is equal to a multiple of 3 plus 1.

This completes the proof. □

Problem 4. **[15 points]** Suppose we want to divide a class of n students into groups each containing either 4 or 5 students.

(a) **[5 points]** Let's try to use strong induction to prove that a class with $n \geq 8$ students can be divided into groups of 4 or 5.

Proof. The proof is by strong induction. Let $P(n)$ be the proposition that a class with n students can be divided into teams of 4 or 5.

Base case: We prove that $P(n)$ is true for $n = 8$, 9, or 10 by showing how to break classes of these sizes into groups of 4 or 5 students:

$$\begin{aligned} 8 &= 4 + 4 \\ 9 &= 4 + 5 \\ 10 &= 5 + 5 \end{aligned}$$

Inductive step: We must show that $P(8), \ldots, P(n)$ imply $P(n+1)$ for all $n \geq 10$. Thus, we assume that $P(8), \ldots, P(n)$ are all true and show how to divide up a class of $n + 1$ students into groups of 4 or 5. We first form one group of 4 students. Then we can divide the remaining $n - 3$ students into groups of 4 or 5 by the assumption $P(n - 3)$. This proves $P(n + 1)$, and so the claim holds by induction. □

This proof contains a critical logical error. Identify the first sentence in the proof that does not follow and explain what went wrong. (Pointing out that the *claim* is false is not sufficient; you must find the first logical error in the *proof*.)

Solution. The first error is in the sentence:

> Then we can divide the remaining $n - 3$ students into groups of 4 or 5 by the assumption $P(n - 3)$.

If $n = 10$, then $P(n - 3) = P(7)$, which is not among our assumptions $P(8), \ldots, P(n)$. In this case, $P(n + 1) = P(11)$ is actually false.

(b) [10 points] Provide a correct strong induction proof that a class with $n \geq 12$ students can be divided into groups of 4 or 5.

Solution. The proof is by strong induction. Let $P(n)$ be the proposition that a class with n students can be divided into teams of 4 or 5.

Base case: We prove that $P(n)$ is true for $n = 12$, 13, 14, and 15 by showing how to break classes of these sizes into groups of 4 or 5 students:

$$\begin{aligned} 12 &= 4 + 4 + 4 \\ 13 &= 4 + 4 + 5 \\ 14 &= 4 + 5 + 5 \\ 15 &= 5 + 5 + 5 \end{aligned}$$

Inductive step: We must show that $P(12), \ldots, P(n)$ imply $P(n+1)$ for all $n \geq 15$. Thus, we assume that $P(12), \ldots, P(n)$ are all true and show how to divide up a class of $n+1$ students. We first form one group of 4 students. Then we can divide the remaining $n-3$ students into groups of 4 or 5 by the assumption $P(n-3)$. (Note that $n \geq 15$ and so $n-3 \geq 12$; thus, $P(n-3)$ is among our assumptions $P(12), \ldots, P(n)$.) This proves $P(n+1)$, and so the claim holds by induction.

Problem 5. [20 points] In the 15-puzzle, there are 15 lettered tiles and a blank square arranged in a 4×4 grid. Any lettered tile adjacent to the blank square can be slid into the blank. For example, a sequence of two moves is illustrated below:

A	B	C	D
E	F	G	H
I	J	K	L
M	O	N	

$\rightarrow$

A	B	C	D
E	F	G	H
I	J	K	L
M	O		**N**

$\rightarrow$

A	B	C	D
E	F	G	H
I	J		L
M	O	**K**	N

In the leftmost configuration shown above, the O and N tiles are out of order. Using only legal moves, is it possible to swap the N and the O, while leaving all the other tiles in their original position and the blank in the bottom right corner? In this problem, you will prove the following theorem stating that the answer is "no".

Theorem. *No sequence of moves transforms the board below on the left into the board below on the right.*

A	B	C	D
E	F	G	H
I	J	K	L
M	**O**	**N**	

A	B	C	D
E	F	G	H
I	J	K	L
M	**N**	**O**	

(a) [2 points] We define the "order" of the tiles in a board to be the sequence of tiles on the board reading from the top row to the bottom row and from left to right within a row. For example, in the right board depicted in the above theorem, the order of the tiles is A, B, C, D, E, etc.

Can a row move change the order of the tiles? Prove your answer.

Solution. No. A row move moves a tile from cell i to cell $i+1$ or vice versa. This tile does not change its order with respect to any other tile. Since no other tile moves, there is no change in the order of any of the other pairs of tiles.

(b) [2 points] How many pairs of tiles will have their relative order changed by a column move? More formally, for how many pairs of letters L_1 and L_2 will L_1 appear earlier in the order of the tiles than L_2 before the column move and later in the order after the column move? Prove your answer correct.

Solution. A column move changes the relative order of exactly three pairs of tiles. Sliding a tile down moves it after the next three tiles in the order. Sliding a tile up moves it before the previous three tiles in the order. Either way, the relative order changes between the moved tile and each of the three it crosses.

(c) [**2 points**] We define an *inversion* to be a pair of letters L_1 and L_2 for which L_1 precedes L_2 in the alphabet, but L_1 appears after L_2 in the order of the tiles. For example, consider the following configuration:

A	B	C	E
D	H	G	F
I	J	K	L
M	N	O	

There are exactly four inversions in the above configuration: E and D, H and G, H and F, and G and F.

What effect does a row move have on the parity of the number of inversions? Prove your answer.

Solution. A row move never changes the parity of the number of inversions. A row move does not change the order of the tiles, so it does not affect the total number of inversions.

(d) [**4 points**] What effect does a column move have on the parity of the number of inversions? Prove your answer.

Solution. A column move always changes the parity of the number of inversions. A column move changes the relative order of exactly three pairs of tiles. An inverted pair becomes uninverted and vice versa. Thus, one exchange flips the total number of inversions to the opposite parity, a second exhange flips it back to the original parity, and a third exchange flips it to the opposite parity again.

(e) [**8 points**] The previous problem part implies that we must make an *odd* number of column moves in order to exchange just one pair of tiles (N and O, say). But this is problematic, because each column move also knocks the blank square up or down one row. So after an *odd* number of column moves, the blank can not possibly be back in the last row, where it belongs! Now we can bundle up all these observations and state an invariant, a property of the puzzle that never changes, no matter how you slide the tiles around.

Lemma. *In every configuration reachable from the position shown below, the parity of the number of inversions is different from the parity of the row containing the blank square.*

row 1	A	B	C	D
row 2	E	F	G	H
row 3	I	J	K	L
row 4	M	O	N	

Prove this lemma.

Solution.

Proof. The proof is by induction. Let $P(n)$ be the proposition that after n moves, the parity of the number of inversions is different from the parity of the row containing the blank square.

Base case: After zero moves, exactly one pair of tiles is inverted (O and N), which is an odd number. And the blank square is in row 4, which is an even number. Therefore, $P(0)$ is true.

Inductive step: Now we must prove that $P(n)$ implies $P(n+1)$ for all $n \geq 0$. So assume that $P(n)$ is true; that is, after n moves the parity of the number of inversions is different from the parity of the row containing the blank square. There are two cases:

1. Suppose move $n+1$ is a row move. Then the parity of the total number of inversions does not change. The parity of the row containing the blank square does not change either, since the blank remains in the same row. Therefore, these two parities are different after $n+1$ moves as well, so $P(n+1)$ is true.
2. Suppose move $n+1$ is a column move. Then the parity of the total number of inversions changes. However, the parity of the row containing the blank square also changes, since the blank moves up or down one row. Thus, the parities remain different after $n+1$ moves, and so $P(n+1)$ is again true.

Thus, $P(n)$ implies $P(n+1)$ for all $n \geq 0$.

By the principle of induction, $P(n)$ is true for all $n \geq 0$. □

(f) [2 points] Prove the theorem that we originally set out to prove.

Solution. In the target configuration on the right, the total number of inversions is zero, which is even, and the blank square is in row 4, which is also even. Therefore, by the lemma, the target configuartion is unreachable.

Problem 6. [15 points]

When exploring the basement of MIT, you find a mysterious room. The room has three desks. On one desk is a red bottle and a blue bottle. On another desk, there is a book. It is open to a page with the following instructions:

Here lies the secret to great fortune, or, if misused, to great peril:

1. *Place 5 copper coins and 3 silver coins in a red bottle. The coins will disappear and 14 gold coins will appear in their place.*
2. *Place 7 gold coins in a blue bottle. The coins will disappear and 3 copper coins and 1 silver coin will appear in their place.*

The third desk has several coins. You count them, and find that there are 100 copper coins, 50 silver coins, and no gold coins.

Let a represent the number of copper coins, b represent the number of silver coins, and c represent the number of gold coins.

(a) [**5 points**] Describe the situation as a state machine, including the transitions.

Solution. Starting state: $(a, b, c) = (100, 50, 0)$

Transitions:

$(a, b, c) \to (a - 5, b - 3, c + 14)$

$(a, b, c) \to (a + 3, b + 1, c - 7)$

(b) [**5 points**] Can there ever be 56 copper coins and 24 silver coins? Prove your answer. (Hint: Look at the expression $a + b$.)

Solution. No. To prove that this is impossible, we will first prove that $a+b$ is always 2 more than a multiple of 4.

Proof. Proof by induction. Let $P(n)$ be the proposition that after any n transitions, the number of copper and silver coins is 2 more than a multiple of 4.

Base case: After 0 transitions, there are 150 copper and silver coins, which is 2 more than a multiple of 4. Thus, $P(0)$ is true.

Inductive step: We assume that $P(n)$ is true, that after any n transitions, the number of copper and silver coins is 2 more than a multiple of 4. Consider a sequence of $n+1$ transitions. Let a_k and b_k be the number of copper and silver coins respectively after the first k transitions. If the last transition uses the red bottle, the number of copper and silver coins $a_{n+1} + b_{n+1}$ is 8 less than the number of copper and silver coins $a_n + b_n$ after the first n transitions. Since we assumed that $a_n + b_n$ is 2 more than a multiple of 4, $a_{n+1} + b_{n+1}$ is 8 less than 2 more than a multiple of 4, so $a_{n+1} + b_{n+1}$ must be 2 more than a multiple of 4. If, on the other hand, the last transition uses the blue bottle, the number of copper and silver coins $a_{n+1} + b_{n+1}$ is 4 more than the number of copper and silver coins $a_n + b_n$ after the first n transitions. Since we assumed that $a_n + b_n$ is 2 more than a multiple of 4, $a_{n+1} + b_{n+1}$ is 4 more than 2 more than a multiple of 4, so $a_{n+1} + b_{n+1}$ must be 2 more than a multiple of 4.

We conclude that the number of copper and silver coins is always 2 more than a multiple of 4. Since $56 + 24 = 80$ is a multiple of 4, it is impossible to end up with 56 copper coins and 24 silver coins.

□

(c) [**5 points**] Is it possible to end up with 270 or more gold coins? Prove your answer. (Hint: Look at the expression $7(a + b) + 4c$.)

Solution. No. To prove that it is impossible, we will first show that it is invariant that $7(a + b) + 4c = 7(100 + 50)$.

Proof. Proof by induction. Let $P(n)$ be the proposition that $7(a + b) + 4c$ is equal to 1050 after any sequence of n transitions.

Base case: After 0 transitions, $7(a+b)+4c = 7(100+50)+0$, which is equal to 1050. Thus, $P(0)$ is true.

Inductive step: We assume that $P(n)$ is true, that after any n transitions, $7(a+b)+4c = 1050$. Consider a sequence of $n+1$ transitions. Let a_k, b_k, and c_k be the number of copper, silver, and gold coins respectively after the first k transitions. If the last transition uses the red bottle, then the number of copper coins a_{n+1} is 5 less than the number of copper coins a_n after the first n transitions, the number of silver coins b_{n+1} is 3 less than the number of silver coins b_n after the first n transitions, and the number of gold coins c_{n+1} is 14 more than the number of gold coins c_n after the first n transitions. So we have that

$$\begin{aligned} 7(a_{n+1}+b_{n+1})+4c_{n+1} &= 7((a_n-5)+(b_n-3))+4(c_n+14) = \\ & 7(a_n+b_n)-56+4c_n+56 = \\ & 7(a_n+b_n)+4c_n = \\ & 1050 \end{aligned}$$

If, on the other hand, the last transition uses the blue bottle, then the number of copper coins a_{n+1} is 3 more than the number of copper coins a_n after the first n transitions, the number of silver coins b_{n+1} is 1 more than the number of silver coins b_n after the first n transitions, and the number of gold coins c_{n+1} is 7 less than the number of gold coins c_n after the first n transitions. So we have that

$$\begin{aligned} 7(a_{n+1}+b_{n+1})+4c_{n+1} &= 7((a_n+3)+(b_n+1))+4(c_n-7) = \\ & 7(a_n+b_n)+28+4c_n-28 = \\ & 7(a_n+b_n)+4c_n = \\ & 1050 \end{aligned}$$

So regardless of whether the red bottle or blue bottle was used last, $7(a+b)+c$ is still equal to 1050. This proves that $P(n) \Rightarrow P(n+1)$, so we conclude that $P(n)$ is true for $n \geq 0$.

If there were at least 270 gold coins, $7(a+b)+4c$ would be at least $4(270) = 1080$, which is greater than 1050. Therefore, it is impossible to end up with at least 270 gold coins.

□