Problem Set 12

Due: Friday, December 5, 7pm

Problem 1. [10 points] Here are seven propositions:

$$\begin{array}{rcrcr} x_1 & \vee & x_3 & \vee & \neg x_7 \\ \neg x_5 & \vee & x_6 & \vee & x_7 \\ x_2 & \vee & \neg x_4 & \vee & x_6 \\ \neg x_4 & \vee & x_5 & \vee & \neg x_7 \\ x_3 & \vee & \neg x_5 & \vee & \neg x_8 \\ x_9 & \vee & \neg x_8 & \vee & x_2 \\ \neg x_3 & \vee & x_9 & \vee & x_4 \end{array}$$

Note that:

1. Each proposition is the OR of three terms of the form x_i or the form $\neg x_i$.
2. The variables in the three terms in each proposition are all different.

Suppose that we assign true/false values to the variables $x_1, \ldots, x_9$ independently and with equal probability.

(a) [5 pts] What is the expected number of true propositions?

(b) [5 pts] Use your answer to prove that there exists an assignment to the variables that makes *all* of the propositions true.

Problem 2. [20 points] MIT students sometimes delay laundry for a few days. Assume all random values described below are mutually independent.

(a) [5 pts] A ***busy*** student must complete 3 problem sets before doing laundry. Each problem set requires 1 day with probability 2/3 and 2 days with probability 1/3. Let B be the number of days a busy student delays laundry. What is $\mathrm{E}[B]$?

Example: If the first problem set requires 1 day and the second and third problem sets each require 2 days, then the student delays for $B = 5$ days.

(b) [5 pts] A ***relaxed*** student rolls a fair, 6-sided die in the morning. If he rolls a 1, then he does his laundry immediately (with zero days of delay). Otherwise, he delays for one day and repeats the experiment the following morning. Let R be the number of days a relaxed student delays laundry. What is $\mathrm{E}[R]$?

Example: If the student rolls a 2 the first morning, a 5 the second morning, and a 1 the third morning, then he delays for $R = 2$ days.

(c) [5 pts] Before doing laundry, an ***unlucky*** student must recover from illness for a number of days equal to the product of the numbers rolled on two fair, 6-sided dice. Let U be the expected number of days an unlucky student delays laundry. What is $\mathrm{E}[U]$?

Example: If the rolls are 5 and 3, then the student delays for $U = 15$ days.

(d) [5 pts] A student is ***busy*** with probability 1/2, ***relaxed*** with probability 1/3, and ***unlucky*** with probability 1/6. Let D be the number of days the student delays laundry. What is $\mathrm{E}[D]$?

Problem 3. [20 points] A gambler plays 120 hands of draw poker, 60 hands of black jack,and 20 hands of stud poker per day. He wins a hand of draw poker withprobability 1/6, a hand of black jack with probability 1/2, and a hand ofstud poker with probability 1/5. Assume the outcomes of the card games are mutually independent.

(a) [5 pts] What is the expected number of hands the gambler wins in a day?

(b) [5 pts] What is the variance in the number of hands won per day?

(c) [5 pts] What would the Markov bound be on the probability that the gambler will win 108 hands on a given day?

(d) [5 pts] What would the Chebyshev bound be on the probability that the gambler will win 108 hands on a given day?

Problem 4. [10 points] Prove that for any random variable, R, and constant, b,

(a) [5 pts] if R and S are independent, then so are $(R + b)$ and S.

(b) [5 pts] Prove that $\mathrm{Var}[R] = 0$ iff R is a constant with probability 1.

Problem 5. [15 points] A man has a set of n keys, one of which fits the door to his apartment. He tries the keys until he finds the correct one. Give the expectation and variance for the number of trials until success if he tries the keys at random (possibly repeating a key tried earlier)

Problem 6. [10 points] Ike Harmon wants to get married, but he isn't sure that he's met his soulmate yet. He decides on the following strategy. He will marry the first woman he meets. Then, if he meets someone he finds more suitable, he will divorce his current wife and marry the newcomer.

Suppose Ike meets a total of n women throughout his life, and for any two women, Ike prefers one to the other. Ike's preference relation is a total order, that is, it is transitive. Due to the randomness in everyday life, Ike unfortunately meets these n women in random order.

Prove that the expected number of times Ike has to marry is $\sim \ln n$.

Hint: Let M_i be the indicator variable for the event that Ike marries the ith woman.

Problem 7. [15 points] Let X, X_1, ..., X_n be independent identically distributed random variables. Define the random variable $S = \sum_{i=1}^{n} X_i$. Define the function $M_Y(s) = \mathrm{E}\left[e^{sY}\right]$ for random variables Y.

(a) [5 pts] Prove $\Pr\{S \geq na\} \leq (M_X(s)e^{-sa})^n$ for $s > 0$. (Hint: Prove $M_S(s) = (M_X(s))^n$ and use the Markov bound.)

(b) [5 pts] Let X be a binary variable with $\Pr\{X = 0\} = p$ and $\Pr\{X = 1\} = 1 - p$. Let $1 - p < a < 1$. Show that $M_X(s)e^{-sa}$ is minimized for $s = \ln \frac{ap}{(1-a)(1-p)}$ giving

$$\min_{s>0} M_X(s)e^{-sa} = \left(\frac{p}{1-a}\right)^{1-a} \left(\frac{1-p}{a}\right)^{a}.$$

By using part (a), this yields the bound

$$\Pr\{S \geq na\} \leq \left[\left(\frac{p}{1-a}\right)^{1-a} \left(\frac{1-p}{a}\right)^{a}\right]^{n}.$$

(c) [5 pts] Compute the bound of part (b) for $p = 0.05$ and $a = 0.99$ and compare this bound to the Chernoff bound as presented in the lecture notes.