Problem Set 10

Due: Tuesday, November 18

Problem 1. **[15 points]** We're interested in the probability that a randomly chosen poker hand (5 cards from a standard 52-card deck) contains cards from at most two suits.

(a) [5 pts] What is an appropriate sample space to use for this problem? What are the outcomes in the event, $\mathcal{E}$, we are interested in? What are the probabilities of the individual outcomes in this sample space?

(b) [10 pts] What is $\Pr(\mathcal{E})$?

Problem 2. **[20 points]** Prove the following probabilistic identity, referred to as the **Union Bound**. You may assume the theorem that the probability of a union of *disjoint* sets is the sum of their probabilities.

Theorem. *Let $A_1, \ldots, A_n$ be a collection of events on some sample space. Then*

$$\Pr(A_1 \cup A_2 \cup \cdots \cup A_n) \leq \sum_{i=1}^{n} \Pr(A_i).$$

(*Hint: Induction*)

Problem 3. **[20 points]** You are organizing a neighborhood census and instruct your census takers to knock on doors and note the sex of any child that answers the knock. Assume that there are two children in a household and that girls and boys are equally likely to be children and to open the door.

A sample space for this experiment has outcomes that are triples whose first element is either B or G for the sex of the elder child, likewise for the second element and the sex of the younger child, and whose third coordinate is E or Y indicating whether the *e*lder child or *y*ounger child opened the door. For example, (B, G, Y) is the outcome that the elder child is a boy, the younger child is a girl, and the girl opened the door.

(a) [5 pts] Let T be the event that the household has two girls, and O be the event that a girl opened the door. List the outcomes in T and O.

(b) [5 pts] What is the probability $\Pr(T \mid O)$, that both children are girls, given that a girl opened the door?

(c) [10 pts] Where is the mistake in the following argument for computing $\Pr(T \mid O)$?

If a girl opens the door, then we know that there is at least one girl in the household. The probability that there is at least one girl is

$$1 - \Pr(\text{both children are boys}) = 1 - (1/2 \times 1/2) = 3/4.$$

So,

$$\begin{aligned}
&\Pr(T \mid \text{there is at least one girl in the household}) \\
&\quad = \frac{\Pr(T \cap \text{there is at least one girl in the household})}{\Pr(\text{there is at least one girl in the household})} \\
&\quad = \frac{\Pr(T)}{\Pr(\text{there is at least one girl in the household})} \\
&\quad = (1/4)/(3/4) = 1/3.
\end{aligned}$$

Therefore, given that a girl opened the door, the probability that there are two girls in the household is 1/3.

Problem 4. [25 points] Let's play a game! We repeatedly flip a fair coin. You have the sequence HHT, and I have the sequence HTT. If your sequence comes up first, then you win. If my sequence comes up first, then I win. For example, if the sequence of tosses is:

$$TTHTHT\underline{HHT}$$

then you win. What is the probability that you win? It may come as a surprise that the answer is very different from 1/2.

This problem is tricky, because the game could go on for an arbitrarily long time. Draw enough of the tree diagram to see a pattern, and then sum up the probabilities of the (infinitely many) outcomes in which you win.

It turns out that for any sequence of three flips, there is another sequence that is likely to come up before it. So there is no sequence which turns up earliest! ... and given any sequence, knowing how to pick another sequence that comes up sooner more than half the time gives you a nice chance to fool people gambling at a bar :-)

Problem 5. [20 points] Professor Plum, Mr. Green, and Miss Scarlet are all plotting to shoot Colonel Mustard. If one of these three has both an *opportunity* and the *revolver*, then that person shoots Colonel Mustard. Otherwise, Colonel Mustard escapes. Exactly one of the three has an *opportunity* with the following probabilities:

$$\begin{aligned}
\Pr(\text{Plum has opportunity}) &= 1/6 \\
\Pr(\text{Green has opportunity}) &= 2/6 \\
\Pr(\text{Scarlet has opportunity}) &= 3/6
\end{aligned}$$

Exactly one has the *revolver* with the following probabilities, regardless of who has an opportuntity:

$$\Pr(\text{Plum has revolver}) = 4/8$$
$$\Pr(\text{Green has revolver}) = 3/8$$
$$\Pr(\text{Scarlet has revolver}) = 1/8$$

(a) [5 pts] Draw a tree diagram for this problem. Indicate edge and outcome probabilities.

(b) [5 pts] What is the probability that Colonel Mustard is shot?

(c) [5 pts] What is the probability that Colonel Mustard is shot, given that Miss Scarlet does not have the revolver?

(d) [5 pts] What is the probability that Mr. Green had an opportunity, given that Colonel Mustard was shot?