

Problem Set 10 Solutions

Due: Tuesday, November 18

Problem 1. [15 points] We're interested in the probability that a randomly chosen poker hand (5 cards from a standard 52-card deck) contains cards from at most two suits.

(a) [5 pts] What is an appropriate sample space to use for this problem? What are the outcomes in the event, \mathcal{E} , we are interested in? What are the probabilities of the individual outcomes in this sample space?

Solution. The natural sample space to use consists of the $\binom{52}{5}$ possible poker hands. We define \mathcal{E} to be the subset of outcomes in which the 5 cards on the outcome come from at most two suits. The sample space is *uniform*: Each hand is equally likely and comes up with probability $1/\binom{52}{5}$. ■

(b) [10 pts] What is $\Pr(\mathcal{E})$?

Solution. Since the sample space is uniform,

$$\Pr(\mathcal{E}) = \frac{|\mathcal{E}|}{\binom{52}{5}}.$$

We can count the size of \mathcal{E} by cases. There are three cases: five cards of the same suit; four cards of one suit and one of another; and three cards of one suit and two of another.

For five of one suit, there are 4 ways to choose the suit and then $\binom{13}{5}$ ways to choose 5 cards from that suit.

For four of one suit and one of another, there are 4 ways to choose the larger suit and $\binom{13}{4}$ ways to choose 4 cards from that suit. Then there are 3 remaining suits from which to choose 1, and 13 choices for the 1 card of that suit.

Finally, for 3 of one suit and 2 of another, there are 4 ways to choose the suit of 3 and $\binom{13}{3}$ ways to choose cards for that suit, and there are 3 remaining suits to choose for the 2 cards, and $\binom{13}{2}$ choices for the 2 cards of that suit. So the total is

$$4 \cdot \binom{13}{5} + 4 \cdot \binom{13}{4} \cdot 3 \cdot 13 + 4 \cdot \binom{13}{3} \cdot 3 \cdot \binom{13}{2},$$

and the probability of at most two suits is

$$\frac{4 \cdot \binom{13}{5} + 4 \cdot \binom{13}{4} \cdot 3 \cdot 13 + 4 \cdot \binom{13}{3} \cdot 3 \cdot \binom{13}{2}}{\binom{52}{5}} = 88/595 \approx 0.15.$$

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Problem 2. [20 points] Prove the following probabilistic identity, referred to as the **Union Bound**. You may assume the theorem that the probability of a union of *disjoint* sets is the sum of their probabilities.

Theorem. Let A_1, \dots, A_n be a collection of events on some sample space. Then

$$\Pr(A_1 \cup A_2 \cup \dots \cup A_n) \leq \sum_{i=1}^n \Pr(A_i).$$

(Hint: Induction)

Solution. For all $n \geq 1$, let $P(n)$ be the proposition that the claim is true.

Base case: Trivially $\Pr(A_1) \leq \Pr(A_1)$, so $P(1)$ is true.

Induction step: Assume that $P(n)$ is true and show $P(n+1)$ for $n \geq 1$.

$$\begin{aligned} \Pr(A_1 \cup A_2 \cup \dots \cup A_{n+1}) &= \Pr((A_1 \cup A_2 \cup \dots \cup A_n) \cup A_{n+1}) \\ &= \Pr(A_1 \cup A_2 \cup \dots \cup A_n) + \Pr(A_{n+1}) \\ &\quad - \Pr((A_1 \cup A_2 \cup \dots \cup A_n) \cap A_{n+1}) \quad (\text{by Inclusion-Exclusion}) \\ &\leq \Pr(A_1 \cup A_2 \cup \dots \cup A_n) + \Pr(A_{n+1}) \\ &\leq \sum_{i=1}^n \Pr(A_i) + \Pr(A_{n+1}) \quad (\text{by Ind. Hyp.}) \\ &= \sum_{i=1}^{n+1} \Pr(A_i) \end{aligned}$$

Thus $P(n)$ is true and the result follows by induction. ■

Problem 3. [20 points] You are organizing a neighborhood census and instruct your census takers to knock on doors and note the sex of any child that answers the knock. Assume that there are two children in a household and that girls and boys are equally likely to be children and to open the door.

A sample space for this experiment has outcomes that are triples whose first element is either **B** or **G** for the sex of the elder child, likewise for the second element and the sex of the younger child, and whose third coordinate is **E** or **Y** indicating whether the elder child or younger child opened the door. For example, $(\mathbf{B}, \mathbf{G}, \mathbf{Y})$ is the outcome that the elder child is a boy, the younger child is a girl, and the girl opened the door.

(a) [5 pts] Let T be the event that the household has two girls, and O be the event that a girl opened the door. List the outcomes in T and O .

Solution. $T = \{GGE, GGY\}$, $O = \{GGE, GGY, GBE, BGY\}$ ■

(b) [5 pts] What is the probability $\Pr(T | O)$, that both children are girls, given that a girl opened the door?

Solution. $1/2$ ■

(c) [10 pts] Where is the mistake in the following argument for computing $\Pr(T | O)$?

If a girl opens the door, then we know that there is at least one girl in the household. The probability that there is at least one girl is

$$1 - \Pr(\text{both children are boys}) = 1 - (1/2 \times 1/2) = 3/4.$$

So,

$$\begin{aligned} & \Pr(T | \text{there is at least one girl in the household}) \\ &= \frac{\Pr(T \cap \text{there is at least one girl in the household})}{\Pr(\text{there is at least one girl in the household})} \\ &= \frac{\Pr(T)}{\Pr(\text{there is at least one girl in the household})} \\ &= (1/4)/(3/4) = 1/3. \end{aligned}$$

Therefore, given that a girl opened the door, the probability that there are two girls in the household is $1/3$.

Solution. The argument is a correct proof that

$$\Pr(T | \text{there is at least one girl in the household}) = 1/3.$$

The problem is that the event, H , that the household has at least one girl, namely,

$$H = \{GGE, GGY, GBE, GBY, BGE, BGY\},$$

is not equal to the event, O , that a girl opens the door. These two events differ:

$$H - O = \{BGE, GBY\},$$

and their probabilities are different. So the fallacy is in the final conclusion where the value of $\Pr(T | H)$ is taken to be the same as the value $\Pr(T | O)$. Actually, $\Pr(T | O) = 1/2$.

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Problem 4. [25 points] Let's play a game! We repeatedly flip a fair coin. You have the sequence HHT , and I have the sequence HTT . If your sequence comes up first, then you win. If my sequence comes up first, then I win. For example, if the sequence of tosses is:

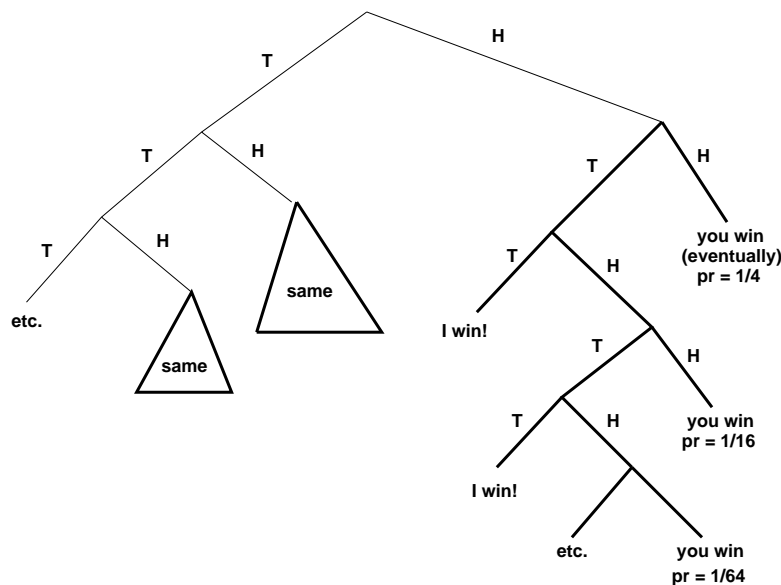
$TTHTHT\underline{HHT}$

then you win. What is the probability that you win? It may come as a surprise that the answer is very different from $1/2$.

This problem is tricky, because the game could go on for an arbitrarily long time. Draw enough of the tree diagram to see a pattern, and then sum up the probabilities of the (infinitely many) outcomes in which you win.

It turns out that for any sequence of three flips, there is another sequence that is likely to come up before it. So there is no sequence which turns up earliest! ...and given any sequence, knowing how to pick another sequence that comes up sooner more than half the time gives you a nice chance to fool people gambling at a bar :-)

Solution. A partial tree diagram is shown below. All edge probabilities are $1/2$.



Let's first focus on the subtree shown in bold. Note that if two heads are flipped in a row, then you are guaranteed to win eventually. The sum of the probabilities of all your winning outcomes in this subtree is:

$$\begin{aligned} \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots &= \frac{1}{4} \cdot \frac{1}{1 - 1/4} \\ &= \frac{1}{3} \end{aligned}$$

The uppermost subtree marked **same** is identical to the one shown in bold, except that each outcome probability is reduced by $1/2$, because it is one edge farther from the root. Thus, the sum of your winning outcomes in this subtree is $1/6$. Similarly, the sum of your winning outcomes in the next subtree marked **same** is $1/12$, and so forth. Overall, your probability of winning is:

$$\begin{aligned} \frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \dots &= \frac{1}{3} \cdot \frac{1}{1 - 1/2} \\ &= \frac{2}{3} \end{aligned}$$

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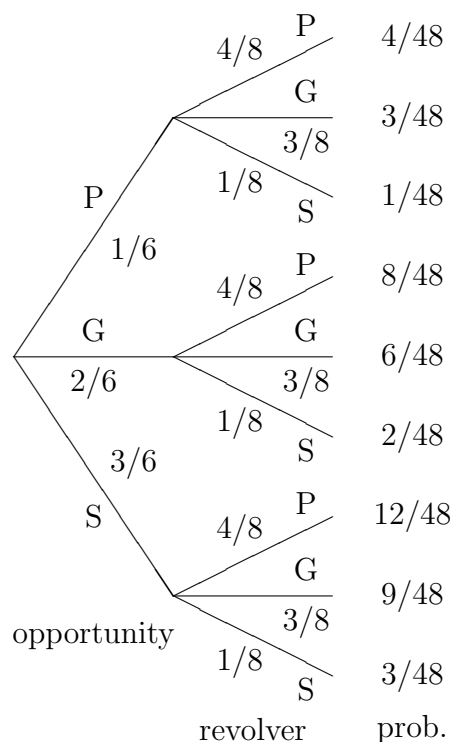
Problem 5. [20 points] Professor Plum, Mr. Green, and Miss Scarlet are all plotting to shoot Colonel Mustard. If one of these three has both an *opportunity* and the *revolver*, then that person shoots Colonel Mustard. Otherwise, Colonel Mustard escapes. Exactly one of the three has an *opportunity* with the following probabilities:

$$\begin{aligned} \Pr(\text{Plum has opportunity}) &= 1/6 \\ \Pr(\text{Green has opportunity}) &= 2/6 \\ \Pr(\text{Scarlet has opportunity}) &= 3/6 \end{aligned}$$

Exactly one has the *revolver* with the following probabilities, regardless of who has an opportunity:

$$\begin{aligned} \Pr(\text{Plum has revolver}) &= 4/8 \\ \Pr(\text{Green has revolver}) &= 3/8 \\ \Pr(\text{Scarlet has revolver}) &= 1/8 \end{aligned}$$

(a) [5 pts] Draw a tree diagram for this problem. Indicate edge and outcome probabilities.



Solution.

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(b) [5 pts] What is the probability that Colonel Mustard is shot?

Solution. Denote each outcome with a pair indicating who has the opportunity and who has the revolver. In this notation, the event that Colonel Mustard is shot consists of all outcomes where a single person has both:

$$\{(P, P), (G, G), (S, S)\}$$

The probability of this event is the sum of the outcome probabilities:

$$\begin{aligned} \Pr(\{(P, P), (G, G), (S, S)\}) &= \Pr((P, P)) + \Pr((G, G)) + \Pr((S, S)) \\ &= 4/48 + 6/48 + 3/48 \\ &= 13/48 \end{aligned}$$

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(c) [5 pts] What is the probability that Colonel Mustard is shot, given that Miss Scarlet does not have the revolver?

Solution. Let S be the event that Colonel Mustard is shot, and let N be the event that

Miss Scarlet does *not* have the revolver. The solution is:

$$\begin{aligned}
 \Pr(S \mid N) &= \frac{\Pr(S \cap N)}{\Pr(N)} \\
 &= \frac{\Pr((P, P), (G, G))}{\Pr((P, P), (P, G), (G, P), (G, G), (S, P), (S, G))} \\
 &= \frac{\frac{4}{48} + \frac{6}{48}}{\frac{4}{48} + \frac{3}{48} + \frac{8}{48} + \frac{6}{48} + \frac{12}{48} + \frac{9}{48}} \\
 &= \frac{5}{21}
 \end{aligned}$$

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(d) [5 pts] What is the probability that Mr. Green had an opportunity, given that Colonel Mustard was shot?

Solution. Let G be the event that Mr. Green has an opportunity, and again let S be the event that Colonel Mustard is shot. Then the solution is:

$$\begin{aligned}
 \Pr(G \mid S) &= \frac{\Pr(G \cap S)}{\Pr(S)} \\
 &= \frac{\Pr((G, G))}{\Pr((P, P), (G, G), (S, S))} \\
 &= \frac{\frac{6}{48}}{\frac{4}{48} + \frac{6}{48} + \frac{3}{48}} \\
 &= \frac{6}{13}
 \end{aligned}$$

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