

## Problem Set 1

**Due:** Monday, September 8

**Problem 1. [12 points]** Let  $t$  be a positive integer. Prove the following statement by proving its contrapositive: if  $r$  is irrational, then  $r^{1/t}$  is irrational. (Be sure to *state* the contrapositive explicitly.)

**Problem 2. [24 points]** In this problem  $x$ ,  $y$ , and  $z$  always denote positive integers. Suppose that  $x^2 + y^2 = z^2$ .

- (a) [8 points] Prove the proposition: if  $z$  is even, then both  $x$  and  $y$  are even.
- (b) [8 points] Prove the proposition:  $x$  and  $y$  cannot both be odd.
- (c) [8 points] Prove the proposition: if at least one of  $x$ ,  $y$ , and  $z$  is odd, then  $z$  is odd.

**Problem 3. [12 points]** A *triangle* is a set of three people such that either every pair has shaken hands or no pair has shaken hands. Prove that among every six people there is a triangle.

(Suggestion: Initially, break the problem into two cases: For any person  $X$  in a given group of six people,

1. There exist at least three other people who shook hands with  $X$ .
2. There exist at least three other people who didn't shake hands with  $X$ .

Why must exactly one of these conditions hold?)

**Problem 4. [16 points]**

- (a) [8 points] Prove that

$$\exists z. [P(z) \wedge Q(z)] \Rightarrow [\exists x. P(x) \wedge \exists y. Q(y)] \quad (1)$$

is valid. (Use the proof in the subsection on Validity in Recitation 1 Notes as a guide to writing your own proof here.)

- (b) [8 points] Prove that the converse of (1) is not valid by describing a counter model as in Recitation 1 Notes.

**Problem 5. [24 points]**

The binary logical connectives  $\wedge$  (*and*),  $\vee$  (*or*), and  $\Rightarrow$  (*implies*) appear often in not only computer programs, but also everyday speech. In computer chip designs, however, it is considerably easier to construct these out of another operation, **nand**, which is simpler to represent in a circuit. Here is the truth table for **nand**:

$P$	$Q$	$P \text{ nand } Q$
true	true	false
true	false	true
false	true	true
false	false	true

- (a) [12 points] For each of the following expressions, find an equivalent expression using only **nand** and  $\neg$  (*not*), as well as grouping parentheses to specify the order in which the operations apply. You may use  $A$ ,  $B$ , and the operators any number of times.

- (i)  $A \wedge B$
- (ii)  $A \vee B$
- (iii)  $A \Rightarrow B$

- (b) [4 points] It is actually possible to express each of the above using only **nand**, without needing to use  $\neg$ . Find an equivalent expression for  $(\neg A)$  using only **nand** and grouping parentheses.

- (c) [8 points] The constants **true** and **false** themselves may be expressed using only **nand**. Construct an expression using an arbitrary statement  $A$  and **nand** that evaluates to **true** regardless of whether  $A$  is **true** or **false**. Construct a second expression that always evaluates to **false**. Do not use the constants **true** and **false** themselves in your statements.

**Problem 6. [12 points]** Let  $A$ ,  $B$ , and  $C$  be sets such that  $A \cap B \cap C = \emptyset$ . Prove that:

$$A \cup B \cup C = (A - B) \cup (B - C) \cup (C - A)$$

You are welcome to use a diagram to aid your own reasoning, but your proof must be text. (For guidance on structuring the proof, see pages 64-71 in Cupillari.)