Sums, Approximations, and Asymptotics

1 The Value of an Annuity

If you won the lottery, would you prefer a million dollars today or \$50,000 a year for the rest of your life? This is a question about the value of an annuity. An ***annuity*** is a financial instrument that pays out a fixed amount of money at the beginning of every year for some specified number of years. In particular, an n-year, \$$m$-payment annuity pays m dollars at the start of each year for n years. In some cases, n is finite, but not always. Examples include lottery payouts, student loans, and home mortgages. There are even Wall Street people who specialize in trading annuities.

A key question is what an annuity is actually worth. For many reasons, \$50,000 a year for 20 years is worth much less than a million dollars right now. For example, consider the last \$50,000 installment. If you had that \$50,000 right now, then you could start earning interest, invest the money in the stock market, or just buy something fun. However, if you don't get the \$50,000 for another 20 years, then someone else is earning all the interest or investment profit. Furthermore, prices are likely to gradually rise over the next 20 years, so you when you finally get the money, you won't be able to buy as much. Furthermore, people only live so long; if you were 90 years old, a payout 20 years in the future would be worth next to nothing!

But what if your choice were between \$50,000 a year for 20 years and a *half* million dollars today?

1.1 The Future Value of Money

In order to address such questions, we have to make an assumption about the future value of money. Let's put most of the complications aside and think about this from a simple perspective. Suppose you invested \$1 today at an annual interest rate of $p\%$. Then \$1 today would become $1 + p$ dollars in a year, $(1 + p)^2$ dollars in two years and so forth. A reasonable estimate for p these days is about 6%.

Looked at another way, a dollar paid out a year from now is worth $1/(1+p)$ dollars today, a dollar paid in two years is worth only $1/(1 + p)^2$ today, etc. Now we can work out the value of an annuity that pays m dollars at the start of each year for the next n years:

payments	current value
$\$m$ today	m
$\$m$ in 1 year	$\frac{m}{1+p}$
$\$m$ in 2 years	$\frac{m}{(1+p)^2}$
$\cdots$	$\cdots$
$\$m$ in $n-1$ years	$\frac{m}{(1+p)^{n-1}}$

$$\text{Total current value:}\quad V = \sum_{k=1}^{n} \frac{m}{(1+p)^{k-1}}$$

So to compute the value of the annuity, we need only evaluate this sum. We *could* plug in values for m, n, and p, compute each term explicitly, and then add them all up. However, this particular sum has an equivalent "closed form" that makes the job easier. In general, a ***closed form*** is a mathematical expression that can be evaluated with a fixed number of basic operations (addition, multiplication, exponentiation, etc.) In contrast, evaluating the sum above requires a number of operations proportional to n.

1.2 A Geometric Sum

Our goal is to find a closed form equivalent to:

$$V = \sum_{k=1}^{n} \frac{m}{(1+p)^{k-1}}$$

Let's begin by rewriting the sum:

$$\begin{aligned} V &= \sum_{j=0}^{n-1} \frac{m}{(1+p)^j} && \text{(substitute } j = k-1\text{)} \\ &= m \sum_{j=0}^{n-1} x^j && \text{(where } x = \frac{1}{1+p}\text{)} \end{aligned}$$

The goal of these substitutions is to put the summation into a special form so that we can bash it with a general theorem. In particular, the terms of the sum

$$\sum_{j=0}^{n-1} x^j = 1 + x + x^2 + x^3 + \ldots + x^{n-1}$$

form a ***geometric series***, which means that each term is a constant times the preceding term. (In this case, the constant is x.) And we've already encountered a theorem about geometric sums:

Theorem 1. *For all $n \geq 1$ and all $z \neq 1$:*

$$\sum_{i=0}^{n} z^j = \frac{1 - z^{n+1}}{1 - z}$$

This theorem can be proved by induction, but that proof gives no hint how the formula might be found in the first place. Here is a more insightful derivation based on the ***perturbation method***. First, we let S equal the value of the sum and then "perturb" it by multiplying by z.

$$\begin{array}{rcccccccccccc} S & = & 1 & + & z & + & z^2 & + & \dots & + & z^n & & \\ zS & = & & & z & + & z^2 & + & \dots & + & z^n & + & z^{n+1} \end{array}$$

The difference between the original sum and the perturbed sum is not so great, because there is massive cancellation on the right side:

$$S - zS = 1 - z^{n+1}$$

Now solving for S gives the expression in Theorem 1:

$$S = \frac{1 - z^{n+1}}{1 - z}$$

You can derive a passable number of summation formulas by mimicking the approach used above. We'll look at some other methods for evaluting sums shortly.

1.3 Return of the Annuity Problem

Now we can solve the annuity pricing problem! The value of an annuity that pays m dollars at the start of each year for n years is:

$$\begin{aligned} V &= m \sum_{j=0}^{n-1} x^j \qquad (\text{where } x = \frac{1}{1+p}) \\ &= m \cdot \frac{1 - x^n}{1 - x} \\ &= m \cdot \frac{1 - \left(\frac{1}{1+p}\right)^n}{1 - \left(\frac{1}{1+p}\right)} \end{aligned}$$

We apply Theorem 1 on the second line, and undo the the earlier substitution $x = 1/(1 + p)$ on the last line.

The last expression is a closed form; it can be evaluated with a fixed number of basic operations. For example, what is the real value of a winning lottery ticket that pays \$50,000 per year for 20 years? Plugging in $m = \$50,000$, $n = 20$, and $p = 0.6$ gives $V \approx \$607,906$. The deferred payments are worth more than a half million dollars today, but not by much!

1.4 Infinite Sums

Would you prefer a million dollars today or $50,000 a year *forever*? This might seem like an easy choice— when infinite money is on offer, why worry about inflation?

This is a question about an ***infinite sum***. In general, the value of an infinite sum is defined as the limit of a finite sum as the number of terms goes to infinity:

$$\sum_{k=0}^{\infty} z_k \qquad \text{means} \qquad \lim_{n\to\infty} \sum_{k=0}^{n} z_k$$

So the value of an annuity with an infinite number of payments is given by our previous answer in the limit as n goes to infinity:

$$\begin{aligned} V &= \lim_{n\to\infty} m \cdot \frac{1 - \left(\frac{1}{1+p}\right)^n}{1 - \left(\frac{1}{1+p}\right)} \\ &= m \cdot \frac{1}{1 - \left(\frac{1}{1+p}\right)} \\ &= m \cdot \frac{1+p}{p} \end{aligned}$$

In the second step, notice that the $1/(1+p)^n$ term in the numerator goes to zero in the limit. The third equation follows by simplifying.

Plugging in $m = \$50,000$ and $p = 0.6$ into this formula gives $V \approx \$883,333$. This means that getting $\$50,000$ per year *forever* is still not as good as a million dollars today! Then again, if you had a million dollars today in the bank earning $p = 6\%$ interest, you could take out and spend $\$60,000$ a year forever. So this answer makes some sense.

More generally, we can get a closed form for infinite geometric sums from Theorem 1 by taking a limit.

Corollary 2. *If* $|z| < 1$, *then:*

$$\sum_{i=0}^{\infty} z^i = \frac{1}{1-z}$$

Proof.

$$\begin{aligned} \sum_{i=0}^{\infty} z^i &= \lim_{n\to\infty} \sum_{i=0}^{n} z^i \\ &= \lim_{n\to\infty} \frac{1 - z^{n+1}}{1-z} \\ &= \frac{1}{1-z} \end{aligned}$$

The first equation uses the definition of an infinite limit, and the second uses Theorem 1. In the limit, the term z^{n+1} in the numerator vanishes since $|z| < 1$. □

We now have closed forms for both finite and infinite geometric series. Some examples are given below. In each case, the solution follows immediately from either Theorem 1 (for finite series) or Corollary 2 (for infinite series).

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots=\sum_{i=0}^{\infty}(1/2)^i=\frac{1}{1-(1/2)}=2$$

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots=\sum_{i=0}^{\infty}(-1/2)^i=\frac{1}{1-(-1/2)}=\frac{2}{3}$$

$$1+2+4+8+\ldots+2^{n-1}=\sum_{i=0}^{n-1}2^i=\frac{1-2^n}{1-2}=2^n-1$$

$$1+3+9+27+\ldots+3^{n-1}=\sum_{i=0}^{n-1}3^i=\frac{1-3^n}{1-3}=\frac{3^n-1}{2}$$

Here is a good rule of thumb: *the sum of a geometric series is approximately equal to the term with greatest absolute value.* In the first two examples, the largest term is equal to 1 and the sums are 2 and 2/3, which are both relatively close to 1. In the third example, the sum is about twice the largest term. In the final example, the largest term is 3^{n-1} and the sum is $(3^n-1)/2$, which is 1.5 times greater.

2 Variants of Geometric Sums

You now know everything about geometric series. But one often encounters sums that cannot be transformed by simple variable substitutions to the form $\sum z^i$. A useful way to obtain new summation formulas from old is by *differentiating* or *integrating* with respect to z.

For example, consider the following series:

$$\sum_{i=1}^{n} iz^i = z+2z^2+3z^3+\ldots+nz^n$$

This is not a geometric series, since the ratio between successive terms is not constant. So our formula for the sum of a geometric series cannot be directly applied. But suppose that we differentiate both sides of that equation:

$$\begin{aligned}\frac{d}{dz}\sum_{i=0}^{n} z^i &= \frac{d}{dz}\,\frac{1-z^{n+1}}{1-z}\\ \sum_{i=0}^{n} iz^{i-1} &= \frac{-(n+1)z^n(1-z)-(-1)(1-z^{n+1})}{(1-z)^2}\\ &= \frac{-(n+1)z^n+(n+1)z^{n+1}+1-z^{n+1}}{(1-z)^2}\\ &= \frac{1-(n+1)z^n+nz^{n+1}}{(1-z)^2}\end{aligned}$$

Often differentiating or integrating messes up the exponent of z in every term. In this case, we now have a formula for a sum of the form $\sum iz^{i-1}$, but we want a formula for $\sum iz^i$. The solution is to multiply both sidez by z. Let's bundle up the result as a theorem:

Theorem 3. *For all $n \geq 0$ and all $z \neq 1$:*

$$\sum_{i=0}^{n} iz^i = \frac{z - (n+1)z^{n+1} + nz^{n+2}}{(1-z)^2}$$

If $|z| < 1$, then the sum converges to a finite value even if there are infinitely many terms. Taking the limit as n tends infinity gives:

Corollary 4. *If $|z| < 1$, then:*

$$\sum_{i=0}^{\infty} iz^i = \frac{z}{(1-z)^2}$$

As a consequence, suppose you're offered \$50,000 at the end of this year, \$100,000 at the end of next year, \$150,000 after the following year, and so on. How much is this worth? Surely *this* is infinite money! Let's work out the answer:

$$\begin{aligned} V &= \sum_{i=1}^{\infty} \frac{im}{(1+p)^i} \\ &= m \cdot \frac{\frac{1}{1+p}}{\left(1 - \frac{1}{1+p}\right)^2} \\ &= m \cdot \frac{1+p}{p^2} \end{aligned}$$

The second line follows from Corollary 4. The last step is simplification.

Setting $m = \$50,000$ and $p = 0.06$ gives value of the annuity: \$14,722,222. Intuitively, even though payments increase every year, they increase only *linearly* with time. In contrast, dollars paid out in the future decrease in value *exponentially* with time. As a result, payments in the distant future are almost worthless, so the value of the annuity is still finite!

Integrating a geometric sum gives yet another summation formula. Let's start with the formula for an infinite geometric sum:

$$\sum_{i=0}^{\infty} z^i = \frac{1}{1-z} \qquad (|z| < 1)$$

Now we integrate both sides from 0 to x:

$$\int_0^x \sum_{i=0}^{\infty} z^i \, dz = \int_0^x \frac{1}{1-z} \, dz$$
$$\left. \sum_{i=0}^{\infty} \frac{z^{i+1}}{i+1} \right|_0^x = -\ln(1-z) \, \Big|_0^x$$
$$\sum_{i=0}^{\infty} \frac{x^{i+1}}{i+1} = -\ln(1-x)$$

Reindexing on the left side with the substitution $j = i + 1$ gives the summation formula:

$$\sum_{j=1}^{\infty} \frac{x^j}{j} = -\ln(1-x)$$

You might have seen this before: this is the the Taylor expansion for $-\ln(1-x)$.

3 Sums of Powers

Long ago, in the before-time, we verified the formula:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

The source of this formula is still a mystery! Sure, we can prove this statement by induction, but where did the expression on the right come from in the first place? Even more inexplicable is the summation formula for consecutive squares:

$$\sum_{i=1}^{n} i^2 = \frac{(2n+1)(n+1)n}{6} \tag{1}$$

Here is one way we might discover such formulas. Remember that sums are the discrete cousins of integrals. So we might guess that the sum of a degree-k polynomial is a degree-$(k+1)$ polynomial, just as if we were integrating. If this guess is correct, then

$$\sum_{i=1}^{n} i^2 = an^3 + bn^2 + cn + d$$

for some constants a, b, c, and d. All that remains is to determine these constants. We can do this by plugging a few values for n into the equation above. Each value gives a linear equation in a, b, c, and d. For example:

$$\begin{aligned} n = 0 \quad &\Rightarrow \quad 0 = d \\ n = 1 \quad &\Rightarrow \quad 1 = a + b + c + d \\ n = 2 \quad &\Rightarrow \quad 5 = 8a + 4b + 2c + d \\ n = 3 \quad &\Rightarrow \quad 14 = 27a + 9b + 3c + d \end{aligned}$$

We now have four equations in four unknowns. Solving this system gives $a = 1/3$, $b = 1/2$, $c = 1/6$, and $d = 0$ and so it is tempting to conclude that:

$$\sum_{i=1}^{n} i^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \tag{2}$$

Be careful! This equation is valid only if we were correct in our initial guess at the form of the solution. If we were wrong, then the equation above might not hold for all n! The only way to be sure is to verify this formula with an induction proof. In fact, we did guess correctly; the expressions on the right sides of equations (1) and (2) are equal.

4 Approximating Sums

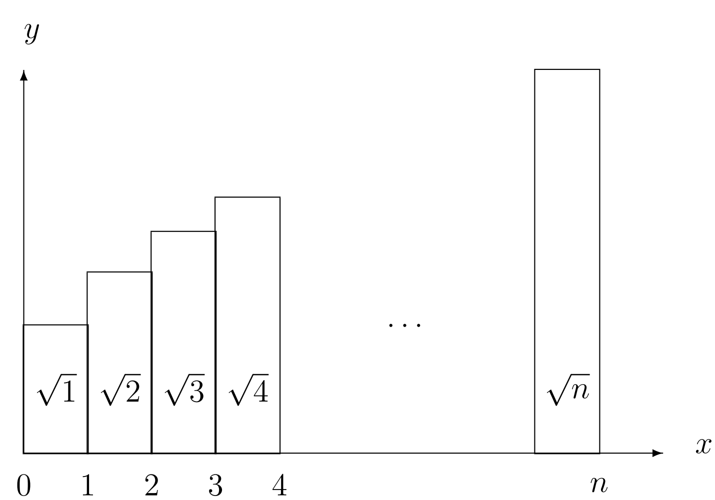
Unfortunately, there is no equivalent closed-form for many of the sums that arise in practice. Here is an example that we'll gnaw on for quite a while:

$$\sum_{i=1}^{n} \sqrt{i} \tag{*}$$

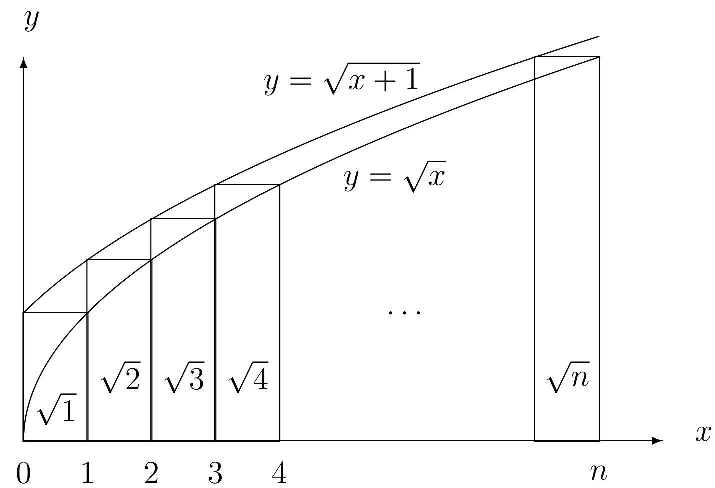
There is no closed form exactly equal to this sum, which we'll refer to as (*) hereafter. However, there do exist good closed-form upper and lower bounds. And for many practical problems, bounds and approximations are good enough.

4.1 Integration Bounds

One way to cope with sums is pass over to the continuous world and apply integration techniques. Suppose that we represent the sum (*) with a "bar graph" on the coordinate plane:



The i-th bar from the left has height $\sqrt{i}$ and width 1. So the area of the i-th bar is equal to the value of the i-th term in the sum. Thus, *the area of the bar graph is equal the value of the sum* (*). Now the graph of the function $y = \sqrt{x+1}$ runs just above the bar graph, and the graph of $y = \sqrt{x}$ runs just below:



So the areas beneath these curves are upper and lower bounds on the area of the bar graph and thus are upper and lower bounds on sum (*). In symbols:

$$\int_0^n \sqrt{x}\,dx \quad \leq \quad \sum_{i=1}^{n} \sqrt{i} \quad \leq \quad \int_0^n \sqrt{x+1}\,dx$$

Now we can obtain closed-form bounds by integration:

$$\left.\frac{x^{3/2}}{3/2}\right|_0^n \quad \leq \quad \sum_{i=1}^{n} \sqrt{i} \quad \leq \quad \left.\frac{(x+1)^{3/2}}{3/2}\right|_0^n$$

$$\frac{2}{3} n^{3/2} \quad \leq \quad \sum_{i=1}^{n} \sqrt{i} \quad \leq \quad \frac{2}{3}\left((n+1)^{3/2} - 1\right)$$

These are pretty good bounds. For example, if $n = 100$, then we get:

$$666.7 \quad \leq \quad \sqrt{1} + \sqrt{2} + \ldots + \sqrt{100} \quad \leq \quad 676.1$$

In order to determine exactly *how good* these bounds are in general, we'll need a better understanding of the rather complicated upper bound, $\frac{2}{3}((n+1)^{3/2} - 1)$.

4.2 Taylor's Theorem

In practice, a simple, approximate answer is often better than an exact, complicated answer. A great way to get such approximations is to dredge Taylor's Theorem up from the dark, murky regions of single-variable calculus.

Theorem 5 (Taylor's Theorem)**.** *If f is a real-valued function, then*

$$f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \underbrace{\frac{f^{(n)}(z)}{n!}x^n}_{\text{“error term”}}$$

for some $z \in [0, x]$, provided f is continuous on $[0, x]$ and n derivatives exist on $(0, x)$.

The idea is to approximate the function f with a short Taylor series. We can then analyze the approximation by studying the "error term". Let's see how this plays out for the function that came up in the previous section:

$$\frac{2}{3}\left((n+1)^{3/2} - 1\right)$$

The crux of this expression is $(n+1)^{3/2}$. However, Taylor's Theorem usually approximates a function $f(x)$ best for x close to zero. But n could be a big number, so we're in some trouble. But we can rewrite the expression into a more tractable form:

$$\frac{2}{3}\left(n^{3/2} \cdot \left(1 + \frac{1}{n}\right)^{3/2} - 1\right)$$

Now the nasty bit is $(1+1/n)^{3/2}$. If we let $x = 1/n$, then our job is to approximate $(1+x)^{3/2}$ where x *is close to zero*. This is a proper target for Taylor's Theorem. First, we compute a few derivatives and their values at 0.

$$\begin{aligned} f(x) &= (1+x)^{3/2} & f(0) &= 1 \\ f'(x) &= \frac{3}{2} \cdot (1+x)^{1/2} & f'(0) &= \frac{3}{2} \\ f''(x) &= \frac{3}{2} \cdot \frac{1}{2} \cdot (1+x)^{-1/2} & f''(0) &= \frac{3}{4} \end{aligned}$$

Now Taylor's Theorem says

$$\begin{aligned} f(x) &= \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(z)}{2!}x^2 \\ &= 1 + \frac{3}{2}x + \frac{3}{8}(1+z)^{-1/2}x^2 \end{aligned}$$

for some $z \in [0, x]$. This expression is maximized when $z = 0$, so we get the upper bound:

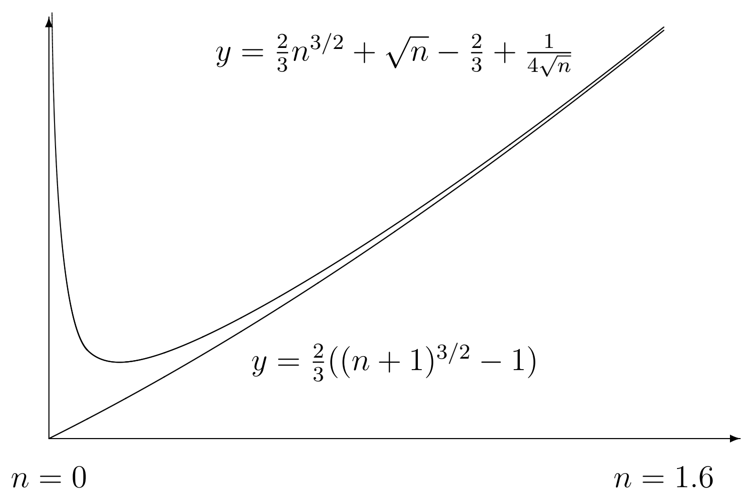
$$(1+x)^{3/2} \leq 1 + \frac{3}{2}x + \frac{3}{8}x^2$$

Now we can upper bound our original expression.

$$\begin{aligned}\frac{2}{3}\left((n+1)^{3/2}-1\right) &= \frac{2}{3}\left(n^{3/2}\cdot\left(1+\frac{1}{n}\right)^{3/2}-1\right)\\ &\leq \frac{2}{3}\left(n^{3/2}\cdot\left(1+\frac{3}{2}\cdot\frac{1}{n}+\frac{3}{8}\cdot\frac{1}{n^2}\right)-1\right)\\ &= \frac{2}{3}n^{3/2}+\sqrt{n}-\frac{2}{3}+\frac{1}{4\sqrt{n}}\end{aligned}$$

There are a lot of terms here, so this might not seem like much of an improvement! However, each term has a simple form. And simpler terms are easier to cancel and combine in subsequent calculations.

Applying Taylor's Theorem involved a lot of grungy math, but you can see the result in a picture. This is a plot of the original function compared to the upper bound we just derived:



The picture shows that our upper bound is *extremely* close to the original function, except for very small values of n.

4.3 Back to the Sum

Before we got to yammering on about Taylor's Theorem, we proved bounds on the sum (*):

$$\frac{2}{3}n^{3/2} \quad\leq\quad \sum_{i=1}^{n}\sqrt{i} \quad\leq\quad \frac{2}{3}\left((n+1)^{3/2}-1\right)$$

Let's rewrite the right side using the upper bound we just derived:

$$\frac{2}{3}n^{3/2} \;\leq\; \sum_{i=1}^{n}\sqrt{i} \;\leq\; \frac{2}{3}n^{3/2}+\sqrt{n}-\frac{2}{3}+\frac{1}{4\sqrt{n}}$$

Now the dominant term in both the lower and upper bounds is $\frac{2}{3}n^{2/3}$, so we can compare the two more readily. As you can see, the gap between the bounds is at most $\sqrt{n}$, which is small relative to the sum as a whole. In fact, the gap between upper and lower bounds that we actually observed for $n = 100$ was very nearly $\sqrt{100} = 10$.

$$666.7 \quad \leq \quad \sqrt{1} + \sqrt{2} + \ldots + \sqrt{100} \quad \leq \quad 676.1$$

There is an elaborate system of notation for describing the quality of approximations. One simple convention is:

$$f(n) \sim g(n) \qquad \text{means} \qquad \lim_{n \to \infty} \frac{f(n)}{g(n)} = 1$$

In terms of this notation, we've just shown that:

$$\sum_{i=1}^{n} \sqrt{i} \quad \sim \quad \frac{2}{3} n^{3/2}$$

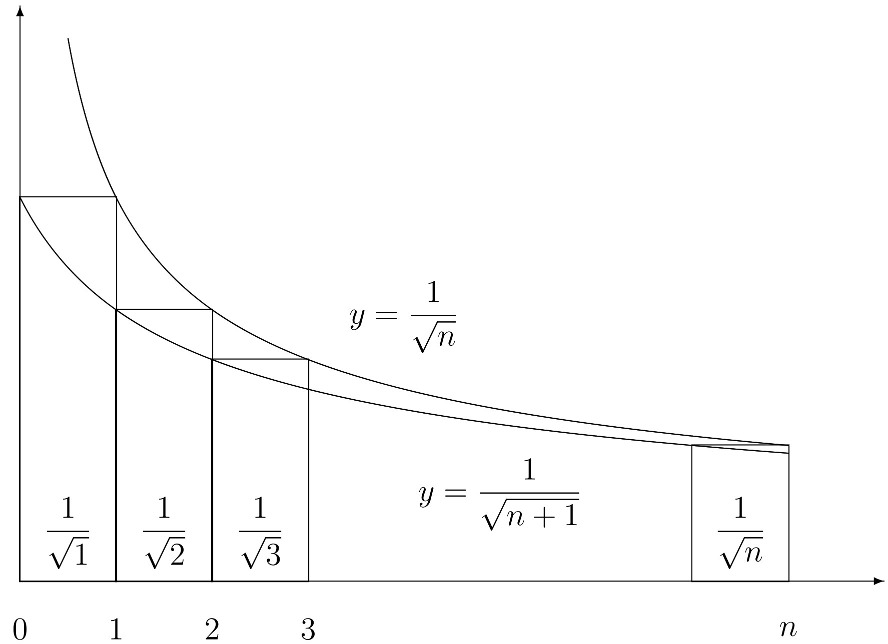
Thus, with a little loss of accuracy, we can replace this vicious sum with a simple closed form.

4.4 Another Integration Example

Let's work through one more example of using integrals to bound a nasty sum. This time our target is:

$$\sum_{i=1}^{n} \frac{1}{\sqrt{i}}$$

As before, we construct a bar graph whose area is equal to the value of the sum. Then we find curves that lie just above and below the bars:



In this way, we get the bounds:

$$\int_0^n \frac{1}{\sqrt{x+1}}\,dx \quad \leq \quad \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \quad \leq \quad \int_0^n \frac{1}{\sqrt{x}}\,dx$$

Evaluating the left-hand side,

$$\int_0^n \frac{1}{\sqrt{x+1}}\,dx = \int_1^{n+1} \frac{1}{\sqrt{x}}\,dx = \left.\frac{\sqrt{x}}{1/2}\right|_1^{n+1} = 2\sqrt{n+1} - 2 \geq 2\sqrt{n} - 2.$$

Evaluating the right-hand side,

$$\int_0^n \frac{1}{\sqrt{x}}\,dx = \left.\frac{\sqrt{x}}{1/2}\right|_0^{n} = 2\sqrt{n}.$$

Thus,

$$2\sqrt{n} - 2 \;\leq\; \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \;\leq\; 2\sqrt{n},$$

which gives us bounds which are within 2 from each other. Note that $\sum_{i=1}^{n} \frac{1}{\sqrt{i}}$ gets arbitrarily large, even though its terms get arbitrarily small.

Sometimes the upper bound is not defined at 0, in which case we can just use $f(1) + \int_1^n f(x)\,dx$ instead. This gives us an easy way to get around a nasty pole at 0.

Now the picture shows that most of the error arises because of the large gap between the two curves for small values of n. In particular, the upper curve goes off to infinity as n approaches zero!

We can eliminate some of this error by summing a couple terms explicitly, and bounding the remaining terms with integrals. For example, summing two terms explicitly gives:

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \int_2^n \frac{1}{\sqrt{x+1}}\, dx \qquad \leq \qquad \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \qquad \leq \qquad \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \int_2^n \frac{1}{\sqrt{x}}\, dx$$

Evaluating the integrals and simplifying, we find:

$$\frac{2+\sqrt{2}-4\sqrt{3}}{2} + 2\sqrt{n+1} \qquad \leq \qquad \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \qquad \leq \qquad \frac{2-3\sqrt{2}}{2} + 2\sqrt{n}$$

Computing the values of the constants gives:

$$2\sqrt{n+1} - 1.75 \qquad \leq \qquad \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \qquad \leq \qquad 2\sqrt{n} - 1.12$$

These bounds are quite tight; they always differ by less than 1. So once again, we've gotten a good handle on a sum that has no exact closed-form equivalent.

4.5 Bounds That Are Simpler to Remember

While everything that was explained in sections 4.1 - 4.4 is true, this year in class, we covered simpler bounds for sums using integration bounds, which we will use going forward. In particular, when f is a positive increasing function of i, we showed that

$$f(1) + \int_1^n f(x)dx \leq \sum_{i=1}^{n} f(i) \leq f(n) + \int_1^n f(x)dx$$

And when f is a positive decreasing function of i, we get just the reverse, namely that

$$f(n) + \int_1^n f(x)dx \leq \sum_{i=1}^{n} f(i) \leq f(1) + \int_1^n f(x)dx$$

These bounds are much simpler to use when $f(0)$ is undefined, and they will generally be a little better and easier to compute that the bounds covered in sections 4.1 - 4.4.