

Not-for-Credit Challenge Problems

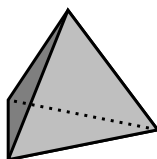
Problem 1. [10 points] The harmonic series has a deranged cousin known as the *Haromoni@* series, which we would probably be better off not paying much attention to:

$$H@(n) = \sum_{k=1}^n \frac{(-1)^k}{k} \quad (1)$$

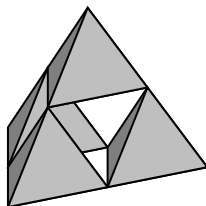
The *Haromoni@* series has its ups and downs, but overall seems to be heading more in the negative direction as n increases. The question is, as $n \rightarrow \infty$, will this series ever converge to some absolute bottom limit, and if so, what value does it converge to?

(Hint: Find the Taylor series for $f(x) = \ln(x + 1)$ in the neighborhood of 0.)

Papproblem 2. [20 papoints] The *Papancake Papacking Paproblem* is a well-known¹ culinary conundrum that has confused countless connoisseurs of carefully crafted breakfast foods over the centuries. The problem involves a curious item called the *papancake*². Unlike normal pancakes, which are generally flat, round, and floppy, papancakes are not flat, not round, and definitely not floppy. In fact, they have the shape of regular tetrahedra (triangular pyramids with equal-length edges):



Over the ages, breakfast chefs have discovered an interesting property of papancakes. Four papancakes can be arranged, corner to corner, to construct a larger papancake with empty space in the middle:



¹By which we mean, we made it up last week.

²One may suppose that the term is short for *pyramidal pancake*, perhaps, but one would be wrong.

This technique is called *papacking*, in contrast to packing, which is squeezing a set of shapes into as small a space as possible. Since the special type of batter required to produce papancakes (processed from the papancake paplant) is extremely expensive to extract, experts often exploit this property to save on the costs of batter while still serving seemingly sizeable papancakes. The Papancake Papacking Papproblem, then, is to determine the minimum amount of batter needed to make a papancake of any given size.

Let P_0 be the basic papancake with no empty space in the middle. It has an edge length of a and a volume of $\sqrt{\frac{1}{72}}a^3$.

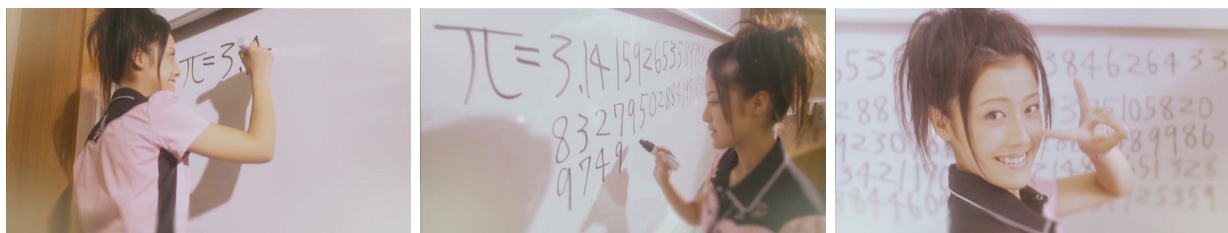
Because of the way larger papancakes are constructed, we can define P_n to be the papancake constructed from four instances of P_{n-1} , for all $n \geq 1$. The edge length of P_n is thus $2^n a$, and the papancake is contained within an enclosing tetrahedron of volume $\sqrt{\frac{1}{72}}2^{3n}a^3$.

Let us define $F(n)$ to be the total amount of empty space in P_n divided by the volume of the enclosing tetrahedron. $F(n)$ is thus a ratio of empty space to total possible volume and can range between 0 and 1.

- (a) [5 pts] Give numerical values for $F(0)$, $F(1)$, \dots , $F(4)$.
- (b) [5 pts] Express $F(n)$ as a sum of n terms.
- (c) [5 pts] What is the limit of $F(n)$ as $n \rightarrow \infty$?
- (d) [5 pts] Prove that if a can be made arbitrarily small, then given any amount of batter ϵ , it is possible to construct a papancake with a total enclosing volume of 1.

Problem 3. [π points]

Singer Miyabi Natsuyuki of the Japanese pop group Buono! can be observed indulging in extreme geekitude in one segment of the music video for Buono!'s October 2007 single "Honto no Jibun" (or "True Self" in English). A few frames of this segment are shown below:



Miyabi has found a series that converges to π :

$$\sum_{n=0}^{\infty} \frac{d(n)}{10^n}, \quad (2)$$

where $d(n)$ is the n th digit of π following the decimal (the initial 3 is the 0th digit).³ The problem with this formula is that one must know the digit sequence $d(n)$ in advance in order to compute the sum, and alas, Miyabi can only remember the first 145 digits. Too bad.

Not to be discouraged, Miyabi decides to consider other possible series, and after experimenting with a few, discovers that the following series seems to oscillate around the value of π :

$$\frac{4}{1} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \dots \quad (3)$$

Does this series actually evaluate to π ?

(a) [5 pts] Miyabi knows that π often shows up in trigonometric formulas and wonders whether it would be possible to relate trigonometry and series. She recalls the derivative of the arctangent from single-variable calculus:

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2} \quad (4)$$

Can you express the right-hand side of the equation as a sum? (*Hint: This looks similar to the closed form for a series we saw in lecture. Make a variable substitution to get it into the right form.*)

(b) [5 pts] Once we have an equivalent sum, we can integrate each term over the interval from 0 to z . Doing the same to the left-hand side, we get $\arctan z$. Suggest a possible value for z so that our integrated terms are of the same form as (3).

(c) [5 pts] Miyabi points out that you've been a bit hasty in deriving this formula, and that your value for z is problematic. Why?

(d) [10 pts] Show how you can fix the problem so that your choice of z works anyway. (*Hint: Express $1/(1+x^2)$ as the sum of a partial sum (up to n) and the difference between $1/(1+x^2)$ and the partial sum. Show that upon integrating and taking the limit as $n \rightarrow \infty$, the difference goes to 0.*)

Problem 4. [30 points]

The enterprising music producer Knutsu[👉], head of the moderately successful pop super-collective LOL! Project, has thought of a brilliant new idea for a pop group. The group, tentatively named Morphing Muffinmix_o, is to operate on a cyclical schedule: at the end of every month, Knutsu[👉] announces that some number of new members (possibly none) have joined, while a subset of the existing members suddenly announce their “graduation” from the group and from LOL! Project in order to pursue academic studies full-time. Each cycle establishes a new “generation” of Morphing Muffinmix_o.

³This formula is not as trivial as it might seem. The notation for a decimal expansion of a number is in fact shorthand for a series that converges to that number.

Knutsu[👉] decides to follow a nondeterministic algorithm in order to determine the number of members in each generation, and defines a value $G(n)$ to represent the number of members in the generation created at the beginning of the n th month (the actual number is $\lfloor G(n) \rfloor$ to avoid dealing with non-integer values, which might otherwise be a bit problematic for those involved). $G(n)$ is initially set to 1 for $n = 1$, and at every monthly iteration, Knutsu[👉] chooses one of two ways the value can change—either multiplying it by a factor dependent on n , or resetting it to 1:

$$G(n+1) = G(n) * e^{1/n}$$

$$G(n+1) = 1$$

Knutsu[👉] is concerned about the order of growth of $G(n)$ and would like our assistance.

(a) [5 pts] The group should not grow too quickly, for Knutsu[👉] is concerned about possibly being possessed by mysterious forces and compelled to make unwise decisions. Give an upper bound $u(n)$ such that $G(n) = O(u(n))$ for any sequence of operations and $G(n) = \Theta(u(n))$ for at least one specific sequence.

(b) [5 pts] Knutsu[👉] is also concerned about possibly making a decision that results in a group with no remaining members. Is this ever possible? Give a lower bound $l(n)$ such that $G(n) = \Omega(l(n))$ for any sequence of operations and $G(n) = \Theta(l(n))$ for at least one specific sequence.

(c) [15 pts] Knutsu[👉] finds both of the bounds to be a bit extreme and would like $G(n)$ to grow at some rate in between. Give a sequence of operations such that there exist functions $f(n)$ and $g(n)$ for which the following relations hold:

$$g(n) = o(u(n))$$

$$f(n) = o(g(n))$$

$$l(n) = o(f(n))$$

$$G(n) = O(g(n))$$

$$G(n) \neq o(f(n))$$

(Hint: Consider the sequence defined by $s_k = 2^{(2^k)}$, and apply a reset operation for every n equal to an element in this sequence.)

(d) [5 pts] Explain why you cannot satisfy $G(n) = \Omega(f(n))$ as well.

Problem 5. [15 points]

Knutsu[👉] thinks this whole “generations” thing is a great idea and would like to extend it to all of the groups in LOL! Project, using a separate function $G(n)$ for each one. The

following are a subset of the groups in LOL! Project and their corresponding $G(n)$:

| | |
|---|---|
| Aaaagghhh!!! | $G(n) = 17$ |
| Celeryz Workshop | $G(n) = n^{n+1/2}e^{-n}$ |
| °C-ube | $G(n) = n!$ |
| DEF.DP | $G(n) = \sum_{k=0}^{\lfloor \ln n \rfloor} e^k$ |
| High-Pass | $G(n) = (n - 1)!$ |
| Pico Moni. | $G(n) = 2 + \cos\left(\frac{\pi n}{2}\right)$ |
| $ \mathbf{v} \cdot \mathbf{u} \cdot \sin \theta$ | $G(n) = \frac{n^2 + 1}{n^2 - 2n + 3}$ |
| W^+ | $G(n) = 1 + \cos\left(\frac{\pi n}{2}\right)$ |

Knutsu 🐱 wants to have some variety in LOL! Project and thinks dividing the groups into equivalence classes by order of growth would give an indication of how diverse the groups are.

(a) [5 pts] Prove that the relation $R = \{(a, b) \mid G_a(n) = \Theta(G_b(n))\}$, where $G_k(n)$ is the function $G(n)$ associated with group k , is an equivalence relation.

(b) [10 pts] Of the eight groups given, how many distinct equivalence classes are there under this relation? Which groups are in which equivalence classes? Justify your answer.