

## Notes for Recitation 21

### 1 Conditional Expectation and Total Expectation

There are conditional expectations, just as there are conditional probabilities. If  $R$  is a random variable and  $E$  is an event, then the conditional expectation  $\text{Ex}(R | E)$  is defined by:

$$\text{Ex}(R | E) = \sum_{w \in S} R(w) \cdot \Pr(w | E)$$

For example, let  $R$  be the number that comes up on a roll of a fair die, and let  $E$  be the event that the number is even. Let's compute  $\text{Ex}(R | E)$ , the expected value of a die roll, given that the result is even.

$$\begin{aligned} \text{Ex}(R | E) &= \sum_{w \in \{1, \dots, 6\}} R(w) \cdot \Pr(w | E) \\ &= 1 \cdot 0 + 2 \cdot \frac{1}{3} + 3 \cdot 0 + 4 \cdot \frac{1}{3} + 5 \cdot 0 + 6 \cdot \frac{1}{3} \\ &= 4 \end{aligned}$$

It helps to note that the conditional expectation,  $\text{Ex}(R | E)$  is simply the expectation of  $R$  with respect to the probability measure  $\Pr_E(\cdot)$  defined in PSet 10. So it's linear:

$$\text{Ex}(R_1 + R_2 | E) = \text{Ex}(R_1 | E) + \text{Ex}(R_2 | E).$$

Conditional expectation is really useful for breaking down the calculation of an expectation into cases. The breakdown is justified by an analogue to the Total Probability Theorem:

**Theorem 1 (Total Expectation).** *Let  $E_1, \dots, E_n$  be events that partition the sample space and all have nonzero probabilities. If  $R$  is a random variable, then:*

$$\text{Ex}(R) = \text{Ex}(R | E_1) \cdot \Pr(E_1) + \dots + \text{Ex}(R | E_n) \cdot \Pr(E_n)$$

For example, let  $R$  be the number that comes up on a fair die and  $E$  be the event that result is even, as before. Then  $\bar{E}$  is the event that the result is odd. So the Total Expectation theorem says:

$$\underbrace{\text{Ex}(R)}_{= 7/2} = \underbrace{\text{Ex}(R | E)}_{= 4} \cdot \underbrace{\Pr(E)}_{= 1/2} + \underbrace{\text{Ex}(R | \bar{E})}_{= ?} \cdot \underbrace{\Pr(\bar{E})}_{= 1/2}$$

The only quantity here that we don't already know is  $\text{Ex}(R | \bar{E})$ , which is the expected die roll, given that the result is odd. Solving this equation for this unknown, we conclude that  $\text{Ex}(R | \bar{E}) = 3$ .

To prove the Total Expectation Theorem, we begin with a Lemma.

**Lemma.** *Let  $R$  be a random variable,  $E$  be an event with positive probability, and  $I_E$  be the indicator variable for  $E$ . Then*

$$\text{Ex}(R | E) = \frac{\text{Ex}(R \cdot I_E)}{\Pr(E)} \quad (1)$$

*Proof.* Note that for any outcome,  $s$ , in the sample space,

$$\Pr(\{s\} \cap E) = \begin{cases} 0 & \text{if } I_E(s) = 0, \\ \Pr(s) & \text{if } I_E(s) = 1, \end{cases}$$

and so

$$\Pr(\{s\} \cap E) = I_E(s) \cdot \Pr(s). \quad (2)$$

Now,

$$\begin{aligned} \text{Ex}(R | E) &= \sum_{s \in S} R(s) \cdot \Pr(\{s\} | E) && \text{(Def of Ex}(\cdot | E)\text{)} \\ &= \sum_{s \in S} R(s) \cdot \frac{\Pr(\{s\} \cap E)}{\Pr(E)} && \text{(Def of Pr}(\cdot | E)\text{)} \\ &= \sum_{s \in S} R(s) \cdot \frac{I_E(s) \cdot \Pr(s)}{\Pr(E)} && \text{(by (2))} \\ &= \frac{\sum_{s \in S} (R(s) \cdot I_E(s)) \cdot \Pr(s)}{\Pr(E)} \\ &= \frac{\text{Ex}(R \cdot I_E)}{\Pr(E)} && \text{(Def of Ex}(R \cdot I_E)\text{)} \end{aligned}$$

□

Now we prove the Total Expectation Theorem:

*Proof.* Since the  $E_i$ 's partition the sample space,

$$R = \sum_i R \cdot I_{E_i} \quad (3)$$

for any random variable,  $R$ . So

$$\begin{aligned} \mathbb{E}_X(R) &= \mathbb{E}_X\left(\sum_i R \cdot I_{E_i}\right) && \text{(by (??))} \\ &= \sum_i \mathbb{E}_X(R \cdot I_{E_i}) && \text{(linearity of } \mathbb{E}_X(\cdot) \text{)} \\ &= \sum_i \mathbb{E}_X(R \mid E_i) \cdot \Pr(E_i) && \text{(by (??))} \end{aligned}$$

□

**Problem 1.** Final exams in 6.042 are graded according to a rigorous procedure:

- With probability  $\frac{4}{7}$  the exam is graded by a *recitation instructor*, with probability  $\frac{2}{7}$  it is graded by a *lecturer*, and with probability  $\frac{1}{7}$ , it is accidentally dropped behind the radiator and arbitrarily given a score of 84.
- *Recitation instructors* score an exam by scoring each problem individually and then taking the sum.
  - There are ten true/false questions worth 2 points each. For each, full credit is given with probability  $\frac{3}{4}$ , and no credit is given with probability  $\frac{1}{4}$ .
  - There are four questions worth 15 points each. For each, the score is determined by rolling two fair dice, summing the results, and adding 3.
  - The single 20 point question is awarded either 12 or 18 points with equal probability.
- *Lecturers* score an exam by rolling a fair die twice, multiplying the results, and then adding a “general impression” score.
  - With probability  $\frac{4}{10}$ , the general impression score is 40.
  - With probability  $\frac{3}{10}$ , the general impression score is 50.
  - With probability  $\frac{3}{10}$ , the general impression score is 60.

Assume all random choices during the grading process are mutually independent.

(a) What is the expected score on an exam graded by a recitation instructor?

**Solution.** Let  $X$  equal the exam score and  $C$  be the event that the exam is graded by a recitation instructor. We want to calculate  $\text{Ex}(X | C)$ . By linearity of (conditional) expectation, the expected sum of the problem scores is the sum of the expected problem scores. Therefore, we have:

$$\begin{aligned} \text{Ex}(X | C) &= 10 \cdot \text{Ex}(\text{T/F score} | C) + 4 \cdot \text{Ex}(\text{15pt prob score} | C) + \text{Ex}(\text{20pt prob score} | C) \\ &= 10 \cdot \left( \frac{3}{4} \cdot 2 + \frac{1}{4} \cdot 0 \right) + 4 \cdot \left( 2 \cdot \frac{7}{2} + 3 \right) + \left( \frac{1}{2} \cdot 12 + \frac{1}{2} \cdot 18 \right) \\ &= 10 \cdot \frac{3}{2} + 4 \cdot 10 + 15 = 70 \end{aligned}$$

(b) What is the expected score on an exam graded by a lecturer?

**Solution.** Now we want  $\text{Ex}(X | \bar{C})$ , the expected score a lecturer would give. Employing linearity again, we have:

$$\begin{aligned} \text{Ex}(X | \bar{C}) &= \text{Ex}(\text{product of dice} | \bar{C}) \\ &\quad + \text{Ex}(\text{general impression} | \bar{C}) \\ &= \left(\frac{7}{2}\right)^2 && \text{(because the dice are independent)} \\ &\quad + \left(\frac{4}{10} \cdot 40 + \frac{3}{10} \cdot 50 + \frac{3}{10} \cdot 60\right) \\ &= \frac{49}{4} + 49 = 61\frac{1}{4} \end{aligned}$$

(c) What is the expected score on a 6.042 exam?

**Solution.** Let  $X$  equal the true exam score. The Total Expectation Theorem implies:

$$\begin{aligned} \text{Ex}(X) &= \text{Ex}(X | C) \Pr(C) + \text{Ex}(X | \bar{C}) \Pr(\bar{C}) \\ &= 70 \cdot \frac{4}{7} + \left(\frac{49}{4} + 49\right) \cdot \frac{2}{7} + 84 \cdot \frac{1}{7} \\ &= 40 + \left(\frac{7}{2} + 14\right) + 12 = 69\frac{1}{2} \end{aligned}$$

**Problem 2.** Here's yet another fun 6.042 game! You pick a number between 1 and 6. Then you roll three fair, independent dice.

- If your number never comes up, then you lose a dollar.
- If your number comes up once, then you win a dollar.
- If your number comes up twice, then you win two dollars.
- If your number comes up three times, you win *four* dollars!

What is your expected payoff? Is playing this game likely to be profitable for you or not?

**Solution.** Let the random variable  $R$  be the amount of money won or lost by the player in a round. We can compute the expected value of  $R$  as follows:

$$\begin{aligned} \text{Ex}(R) &= -1 \cdot \Pr(0 \text{ matches}) + 1 \cdot \Pr(1 \text{ match}) + 2 \cdot \Pr(2 \text{ matches}) + 4 \cdot \Pr(3 \text{ matches}) \\ &= -1 \cdot \left(\frac{5}{6}\right)^3 + 1 \cdot 3 \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^2 + 2 \cdot 3 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right) + 4 \cdot \left(\frac{1}{6}\right)^3 \\ &= \frac{-125 + 75 + 30 + 4}{216} \\ &= \frac{-16}{216} \end{aligned}$$

You can expect to lose  $16/216$  of a dollar (about 7.4 cents) in every round. This is a horrible game!

**Problem 3.** The number of squares that a piece advances in one turn of the game Monopoly is determined as follows:

- Roll two dice, take the sum of the numbers that come up, and advance that number of squares.
- If you roll *doubles* (that is, the same number comes up on both dice), then you roll a second time, take the sum, and advance that number of additional squares.
- If you roll doubles a second time, then you roll a third time, take the sum, and advance that number of additional squares.
- However, as a special case, if you roll doubles a third time, then you go to jail. Regard this as advancing zero squares overall for the turn.

- (a) What is the expected sum of two dice, given that the same number comes up on both?

**Solution.** There are six equally-probable sums: 2, 4, 6, 8, 10, and 12. Therefore, the expected sum is:

$$\frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 4 + \dots + \frac{1}{6} \cdot 12 = 7$$

- (b) What is the expected sum of two dice, given that different numbers come up? (Use your previous answer and the Total Expectation Theorem.)

**Solution.** Let the random variables  $D_1$  and  $D_2$  be the numbers that come up on the two dice. Let  $E$  be the event that they are equal. The Total Expectation Theorem says:

$$\text{Ex}(D_1 + D_2) = \text{Ex}(D_1 + D_2 \mid E) \cdot \Pr(E) + \text{Ex}(D_2 + D_2 \mid \bar{E}) \cdot \Pr(\bar{E})$$

Two dice are equal with probability  $\Pr(E) = 1/6$ , the expected sum of two independent dice is 7, and we just showed that  $\text{Ex}(D_1 + D_2 \mid E) = 7$ . Substituting in these quantities and solving the equation, we find:

$$7 = 7 \cdot \frac{1}{6} + \text{Ex}(D_2 + D_2 \mid \bar{E}) \cdot \frac{5}{6}$$

$$\text{Ex}(D_2 + D_2 \mid \bar{E}) = 7$$

- (c) To simplify the analysis, suppose that we always roll the dice three times, but may ignore the second or third rolls if we didn't previously get doubles. Let the random variable  $X_i$  be the sum of the dice on the  $i$ -th roll, and let  $E_i$  be the event that the  $i$ -th roll is doubles. Write the expected number of squares a piece advances in these terms.

**Solution.** From the total expectation formula, we get:

$$\begin{aligned} \text{Ex}(\text{advance}) &= \text{Ex}(X_1 | \overline{E_1}) \cdot \Pr(\overline{E_1}) \\ &\quad + \text{Ex}(X_1 + X_2 | E_1 \cap \overline{E_2}) \cdot \Pr(E_1 \cap \overline{E_2}) \\ &\quad + \text{Ex}(X_1 + X_2 + X_3 | E_1 \cap E_2 \cap \overline{E_3}) \cdot \Pr(E_1 \cap E_2 \cap \overline{E_3}) \\ &\quad + \text{Ex}(0 | E_1 \cap E_2 \cap E_3) \cdot \Pr(E_1 \cap E_2 \cap E_3) \end{aligned}$$

Then using linearity of (conditional) expectation, we refine this to

$$\begin{aligned} \text{Ex}(\text{advance}) &= \text{Ex}(X_1 | \overline{E_1}) \cdot \Pr(\overline{E_1}) \\ &\quad + (\text{Ex}(X_1 | E_1 \cap \overline{E_2}) + \text{Ex}(X_2 | E_1 \cap \overline{E_2})) \cdot \Pr(E_1 \cap \overline{E_2}) \\ &\quad + (\text{Ex}(X_1 | E_1 \cap E_2 \cap \overline{E_3}) + \text{Ex}(X_2 | E_1 \cap E_2 \cap \overline{E_3}) + \text{Ex}(X_3 | E_1 \cap E_2 \cap \overline{E_3})) \\ &\quad \cdot \Pr(E_1 \cap E_2 \cap \overline{E_3}) \\ &\quad + 0. \end{aligned}$$

Using mutual independence of the rolls, we simplify this to

$$\begin{aligned} \text{Ex}(\text{advance}) &= \text{Ex}(X_1 | \overline{E_1}) \cdot \Pr(\overline{E_1}) \\ &\quad + (\text{Ex}(X_1 | E_1) + \text{Ex}(X_2 | \overline{E_2})) \cdot \Pr(E_1) \cdot \Pr(\overline{E_2}) \\ &\quad + (\text{Ex}(X_1 | E_1) + \text{Ex}(X_2 | E_2) + \text{Ex}(X_3 | \overline{E_3})) \cdot \Pr(E_1) \cdot \Pr(E_2) \cdot \Pr(\overline{E_3}) \end{aligned} \tag{4}$$

(d) What is the expected number of squares that a piece advances in Monopoly?

**Solution.** We plug the values from parts (a) and (b) into equation (??):

$$\begin{aligned} \text{Ex}(\text{advance}) &= 7 \cdot \frac{5}{6} + (7 + 7) \cdot \frac{1}{6} \cdot \frac{5}{6} + (7 + 7 + 7) \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} \\ &= 8 \frac{19}{72} \end{aligned}$$