Problem Set 9

Due: *Monday November 6 at 8 PM*

Problem 1. [15 points]

(a) Describe a bijection between the sequences $x_1, x_2, \ldots, x_k$ of positive integers such that

$$x_1 < x_2 < \cdots < x_k \leq n \tag{1}$$

and the length n bit-strings (i.e., strings of 0's and 1's) containing exactly k 1's.

(b) Use the bijection to write a closed form (which may involve factorials) for the number of sequences satisfying (1).

Problem 2. [20 points] Every Halloween, 3 children visit your neighborhood trick-or-treating. This year, you have 4 pieces of candy to give them, all of which you will place into 3 bags. You can choose any placement of candy in the bags, including leaving a bag empty (poor kid!). How many placements are there if:

(a) You have 4 different types of candy, and the bags are labeled for the children, Ann, Bob, and Cuauhtemoc?

Note: once the pieces of candy are in a bag, there is no way to tell the order in which they were placed.

(b) You only have 1 type of candy, and the bags are unlabeled because you couldn't figure out how to spell "Ann" properly (i.e. the bags are indistinguishable except for the number of pieces of candy in them)?

(c) You have 4 types of candy, but the bags are unlabeled?

(d) The bags are labeled, but you only have 1 type of candy?

Problem 3. [15 points] An urn contains balls numbered from 1 to 9. You draw 6 balls from the urn. How many different *ordered* draws can you get if:

(a) Ball 1 is included in the draw?

(b) Ball 1 and ball 2 are both included in the draw?

(c) Exactly one of ball 1 and ball 2 is included in the draw?

Problem 4. [10 points] (The two parts of this problem are independent.)

(a) Describe a bijection between the length n bit-strings with an even number of `1`'s and the length n bit-strings with an odd number of `1`'s. Explain why any such bijection leads to a simple formula for the number of even-size subsets of any set of size n.

(b) A shelf holds 12 books in a row. How many ways are there to choose five books so that no two adjacent books are chosen?

Problem 5. [15 points] Suppose we have n distinct points on the sphere, where the sphere is taken to mean all points of the form $(x, y, z) \in \mathcal{R}^3$ with $x^2 + y^2 + z^2 = 1$. Show that there is a closed hemisphere containing $2 + \lceil \frac{n-2}{2} \rceil$ of the n points.

Here, by closed hemisphere we mean exactly half of the sphere, including its boundary points. For instance, if we take two distinct points p, q on the sphere, and draw the unique circle C containing p and q centered at the origin, then the sphere decomposes into two closed hemispheres with common intersection C.

Problem 6. [15 points] We want to split the class into teams of four. Suppose there are $4n$ students in the class, so there should be n teams. Write a simple closed-form formula (which may involve factorials) for the number of ways the class could be split. Explain your answer.

Problem 7. [10 points] In preparation for a 6.042 study session, you want to calculate the number of different ways to make sundaes for you and your friends. You have 10 different toppings, and you want to make four sundaes such that each sundae has between one and four (inclusive) toppings, and you don't reuse any toppings. The sundaes are going to 4 different people, so their order matters! How many ways can this be done?