Problem Set 6

Due: Monday, October 16 at 8pm

Problem 1. [10 points] In this problem we continue the study of planar graphs, as studied in recitation 10. Show that any planar graph can be colored with only 6 colors.

Problem 2. [15 points] In this problem we study some properties of relations. Recall that a relation $R \subseteq X \times Y$ is a set of pairs (x, y).

(a) A function $F \subseteq X \times Y$ is a relation with the extra property that if $(x, y) \in F$ and $(x, y') \in F$, then $y = y'$. Let $Z = \{0, 1, 2, \ldots, p-1\}$. Consider the set S of all pairs $(x, y) \in Z \times Z$ for which $x = y^2 \bmod p$. Prove or disprove: S is a function.

Recall that a special type of relation is an equivalence relation, that is, a relation that is reflexive, symmetric, and transitive. For each of the following, either prove that it is an equivalence relation and state its equivalence classes, or give an example of why it is not an equivalence relation.

(b) $R := \{(x, y) \in \{0,1\}^n \times \{0,1\}^n \mid \Delta(x, y) \leq 1\}$, where $\Delta(x, y)$ denotes the *Hamming distance* of x and y, that is, the number of coordinates which differ. For instance, the strings 000 and 010 have Hamming distance 1 since they differ on the second coordinate, and the strings 011 and 110 have Hamming distance 2 since they differ on the first and last coordinates.

(c) $R := \{(x, y) \in \mathbb{R} \times \mathbb{R}\text{s. t. } \exists n \in \mathbb{Z}, y = 2^n x\}$.

Problem 3. [15 points] In this problem we study partial orders (posets). Recall that a partial order $\preceq$ on a set X is reflexive ($x \preceq x$), anti-symmetric ($x \preceq y \wedge y \preceq x \rightarrow x = y$), and transitive ($x \preceq y \wedge y \preceq z \rightarrow x \preceq z$). Note that it may be the case that neither $x \preceq y$ nor $y \preceq x$. A chain is a list of *distinct* elements $x_1, \ldots, x_i$ in X for which $x_1 \preceq x_2 \preceq \cdots \preceq x_i$. An antichain is a subset S of X such that for all distinct $x, y \in S$, neither $x \preceq y$ nor $y \preceq x$.

The aim of this problem is to show that any sequence of $(n-1)(m-1)+1$ integers either contains a non-decreasing subsequence of length n or a decreasing subsequence of length m. Note that the given sequence may be out of order, so, for instance, it may have the form $1, 5, 3, 2, 4$ if $n = m = 2$. In this case the longest non-decreasing and longest decreasing subsequences have length 3 (for instance, consider $1, 2, 4$ and $5, 3, 2$).

(a) Label the given sequence of $(n-1)(m-1)+1$ integers $a_1, a_2, \ldots, a_{(n-1)(m-1)+1}$. Show the following relation $\preceq$ on $\{1, 2, 3, \ldots, (n-1)(m-1)+1\}$ is a poset: $i \preceq j$ if and only if $i \leq j$ and $a_i \leq a_j$ (as integers)

For the next part, we will need to use Dilworth's theorem, as covered in lecture. Recall that Dilworth's theorem states that if $(X, \preceq)$ is any poset whose longest chain has length n, then X can be partitioned into at most n disjoint antichains.

(b) Show that in any sequence of $(n-1)(m-1)+1$ integers, either there is a non-decreasing subsequence of length n or a decreasing subsequence of length m.

(c) Construct a sequence of $(n-1)(m-1)$ integers, for arbitrary n and m, that has no non-decreasing subsequence of length n and no decreasing subsequence of length m. Thus in general, the result you obtained in the previous part is best-possible.