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Problem Set 1 Solutions

Due: Monday, September 11

Problem 1. [15 points] Prove the following statement by proving its contrapositive: if r is irrational, then $r^{1/3}$ is also irrational. (Be sure to *state* the contrapositive explicitly.)

Solution. We prove the contrapositive: if $r^{1/3}$ is rational, then r is rational. If $r^{1/3}$ is rational, then there exists an integer a and a positive integer b such that:

$$r^{1/3} = \frac{a}{b}$$

Cubing both sides gives:

$$r = \frac{a^3}{b^3}$$

Since a^3 and b^3 are integers, this implies that r is also rational. (Note that $b^3 \neq 0$, since b is positive.)

Problem 2. [15 points] Use proof by contradiction to show that $\log_3 8$ is irrational. Suppose you try to use the same proof outline to show that $\log_2 8$ is irrational, where does it break down?

Solution. We use proof by contradiction. Suppose that $\log_3 8$ is rational. Then there exists an integer a and a positive integer b such that:

$$\log_3 8 = \frac{a}{b}$$

This implies that:

$$8 = 3^{a/b}$$
$$8^b = 3^a$$

(Raising the first equation to the b-th power gives the second.) Now the left side of the last equation is even and the right side is odd. This is a contradiction. Therefore, our supposition was wrong, and $\log_3 8$ is irrational. When attempting to prove that $\log_2 8$ is irrational, both sides of the equation will be even and so there will be no contradiction.

Problem 3. [15 points] A student is trying to prove that propositions p, q, and r are all true. She proceeds as follows. First, she proves three facts: $p \rightarrow q$, $q \rightarrow r$, and $r \rightarrow p$. Then she concludes, "Thus obviously p, q, and r are all true."

(a) Using logic notation and the symbols p, q, and r, write down the logical implication that she uses in her final step.

Solution.

$$((p \rightarrow q) \land (q \rightarrow r) \land (r \rightarrow p)) \rightarrow (p \land q \land r)$$

(b) Use a truth table to determine whether this logical implication is a tautology.
Solution.

p	q	r	$((p \rightarrow q) \land (q \rightarrow r) \land (r \rightarrow p))$	$(p \land q \land r)$	complete expression
T	T	T	T	T	T
T	T	F	F	F	T
T	F	T	F	F	T
T	F	F	F	F	T
F	T	T	F	F	T
F	T	F	F	F	T
F	F	T	F	F	T
F	F	F	T	F	$\rightarrow F \leftarrow$

The truth table indicates that the implication that she uses is *not* a tautology.

(c) Is her proof that propositions p, q, and r are all true correct?

Solution. Her proof is incorrect; she makes use of a false proposition. (However, the first part of her argument is sufficient to show that either all three statements are true *or* all three statements are false.)

Problem 4. [20 points] Express each of the following statements in formal logic notation. In addition to the usual logic symbols and variables, you may build predicates using addition, multiplication, constants, and equality and inequality symbols. The domain of discourse is the non-negative integers. For example, the statement "n is an odd number" could be written "$\exists x(2 \cdot x + 1 = n)$".

(a) n is the sum of three perfect squares.

Solution.

$$\exists x \exists y \exists z(x \cdot x + y \cdot y + z \cdot z = n)$$

(b) n is a prime number.

Solution.

$$(n > 1) \land \neg(\exists x \exists y((x > 1) \land (y > 1) \land (x \cdot y = n))$$

(c) n is a product of exactly two primes.

Solution.

$$\begin{aligned}\exists x \exists y \quad (\quad & (x > 1) \wedge \neg(\exists a \exists b((a > 1) \wedge (b > 1) \wedge (a \cdot b = x))) \wedge \\ & (y > 1) \wedge \neg(\exists c \exists d((c > 1) \wedge (d > 1) \wedge (c \cdot d = y))) \wedge \\ & (x \cdot y = n))\end{aligned}$$

Problem 5. [20 points] A self-proclaimed "great logician" has invented a new quantifier, on par with $\exists$ ("there exists") and $\forall$ ("for all"). The new quantifier is symbolized by $\forall^{(1)}$ and read "for all but one". The proposition $\forall^{(1)} x P(x)$ is true if it is that case that for all but exactly one x in the domain of discourse the predicate $P(x)$ is true. The logician has noted, "There used to be two quantifiers, but now there are three! I have extended the whole field of mathematics by 50%!"

(a) Rewrite the proposition $\forall^{(1)} x P(x)$ using only the $\exists$ quantifier.

Solution. To do problems like this, it is advisable to first write the statement using both $\exists$ and $\forall$ quantifiers, and then convert all occurrences of $\forall$ to $\exists$ using the rule that $\forall x P(x)$ is the same as $\neg \exists x(\neg P(x))$.

In this case the expression is $\exists x(\neg P(x) \wedge (\forall y(y = x \vee P(y))))$.

Converting $\forall$ to $\exists$, we get:

$$\exists x(\neg P(x) \wedge \neg(\exists y(x \neq y \wedge \neg P(y))))$$

(notice that while converting, we used the fact that $\neg(y = x \vee P(y))$ is the same as $(y \neq x \wedge \neg P(y))$).

(b) Rewrite the proposition $\forall^{(1)} x P(x)$ using only the $\forall$ quantifier.

Doing as above, we arrive at the following expression:

Solution.

$$\neg \forall x(P(x) \vee \neg(\forall y(x = y \vee P(y))))$$

Problem 6. [15 points]

Universal generalization is an inference rule so natural that you've probably used it many times without a second thought. It says that if you can prove that $P(x)$ is true for some arbitrary, unspecified $x \in S$, then you can conclude that $P(x)$ is true for all $x \in S$. Use universal generalization to prove that:

$$\forall\, x \in [-1, 1] \quad 1 - x^2 \leq 1 - x^2 + x^4 - x^6$$

Here [-1,1] denotes the set of all real numbers between -1 and 1, including -1 and 1. (If you prove this claim any old way, you'll probably discover in retrospect that you implicitly relied on universal generalization.)

Solution. Let x be an arbitrary, unspecified element of $[-1, 1]$. Then $x^2 \leq 1$, which means $1 - x^2$ is nonnegative. We also know

$$1 \leq 1 + x^4$$

since x^4 is nonnegative. Multiplying both sides of this inequality by $1-x^2$ does not reverse its direction, since we showed that $1 - x^2$ is nonnegative. Thus, we have:

$$\begin{aligned} 1 - x^2 &\leq (1 + x^4)(1 - x^2) \\ &= 1 - x^2 + x^4 - x^6 \end{aligned}$$

Since we proved the inequality for an arbitrary, unspecified element $x \in [-1, 1]$, universal generalization implies that the inequality holds for all $x \in [-1, 1]$.

Note: This may be the last you'll ever hear of universal generalization. We'll continue to use it all the time, but just won't mention it explicitly.