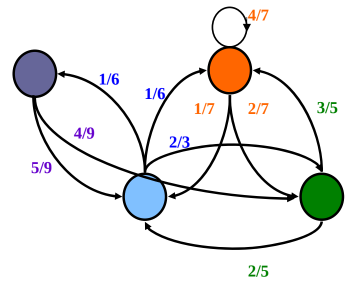
Mathematics for Computer Science

MIT 6.042J/18.062J

Random Walks

Graph with probable transitions



Graph with probable transitions

Questions

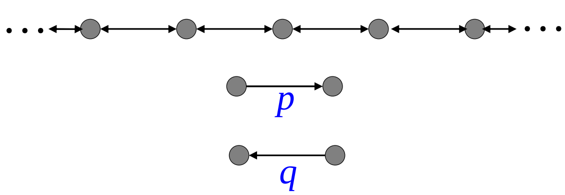
- Pr{blue reaches orange *before* green}
- Pr{blue *ever* reaches orange}
- E[#steps blue to orange]
- Average fraction of time at blue

Random Walks

Applications

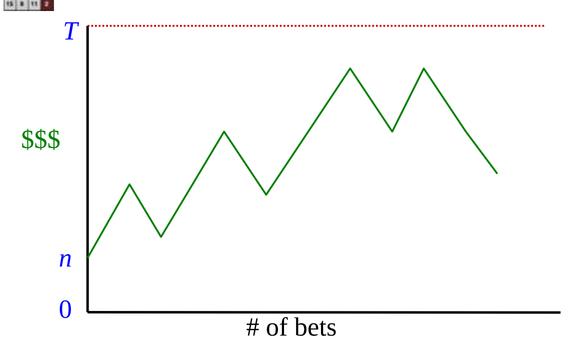
- Finance – Stocks, options
- Algorithms – web search, clustering
- Physics – Brownian Motion

1-Dimensional Walk



Gambler's Ruin

Gambler's Ruin



Gambler's Ruin

Parameters:

- n ::= initial capital (stake)
- T ::= gambler's Target
- p ::= Pr{win \$1 bet}
- q ::= $1 - p$
- m ::= intended profit = $T - n$

Gambler's Ruin

Three general cases:

- Biased against $p < 1/2$
- Biased in favor $p > 1/2$
- Unbiased (Fair) $p = 1/2$

Fair Case: $p = q = 1/2$

Let w ::= Pr{reach Target}

$$E[\$\$] = w\cdot(T-n) + (1-w)\cdot(-n)$$
$$= wT - n$$

But game is *fair*, so E[\$\$ won] = 0

$$w = \frac{n}{T}$$

Fair Case: $p = q = 1/2$

Let w ::= Pr{reach Target}

$$w = \frac{n}{T}$$

Fair Case

Consequences

n=500, T=600

Pr{win \$100} = 500/600 ≈ 0.83

n=1,000,000, T=1,000,100

Pr{win \$100} ≈ 0.9999

Biased Against: $p < 1/2 < q$

Betting red in US roulette

$$p = 18/38 = 9/19 < 1/2$$

Biased Against: p < 1/2 < q

Astonishing Fact!

Pr{win \$100 starting with \$500}

< 1/37,000 !

(was 5/6 in the unbiased case.)

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Biased Against: p < 1/2 < q

More amazing still!

Pr{win \$100 starting with \$1M}

< 1/37,000

Pr{win \$100 starting w/ any \$*n* stake}

< 1/37,000

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Winning in the Unfair Case

Team Problem: for $p < q$,

$$w_n \le \frac{(q/p)^n}{(q/p)^T} = \left(\frac{p}{q}\right)^m$$

where $m ::= T - n$ = intended profit

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Winning in the Unfair Case

for $p < q$:

$$\left(\frac{p}{q}\right)^m$$

is exponentially decreasing in m,

the intended profit.

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Losing in Roulette

$p = 18/38$, $q = 20/38$

$$\Pr\{\text{win } \$100\} = \left(\frac{18/38}{20/38}\right)^{100} = \left(\frac{9}{10}\right)^{100} < \frac{1}{37,648}$$

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Losing in Roulette

$$\Pr\{\text{win } \$200\} = (\Pr\{\text{win } \$100\})^2 = \left(\frac{1}{37,648}\right)^2 < \frac{1}{70,000,000}$$

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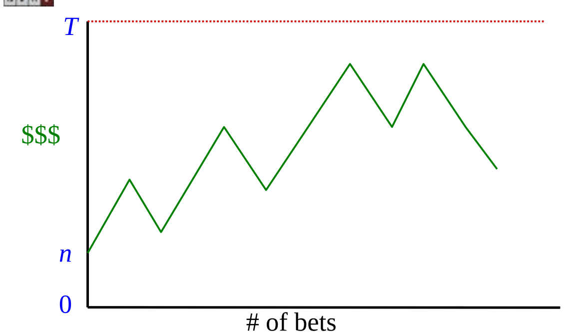

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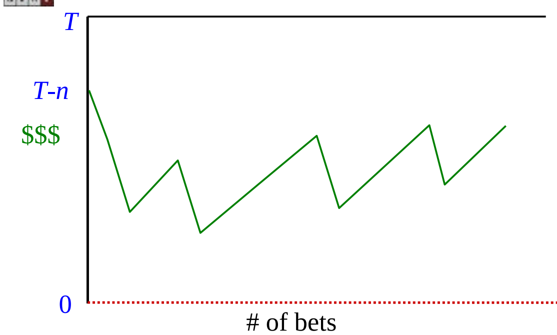
Gambler's Ruin



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Fair Case

$pr\{\text{lose starting with } \$n\}$

$= pr\{\text{win starting with } \$(T - n)\}$

$$= \frac{T-n}{T}$$

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Fair Case for $T = \infty$

$Pr\{\text{lose starting with } n \mid T = \infty\}$

$\geq Pr\{\text{lose starting with } n \mid T < \infty\}$

$= \frac{T-n}{T} \to \infty$ as $T \to \infty$

So if the gambler keeps betting, he is sure to go broke.

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Return to the origin.

If you start at the origin and move left or right with equal probability, and keep moving in this way,

Pr{return to origin} = 1

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How Many Bets?

What is the expected number of bets for the game to end?

– either by winning $\$(T\text{-}n)$ or by going broke (losing $\$n$).

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How Many Bets? Fair Case

E[# bets] = $n(T-n)$ =
(initial stake)·(intended profit)
proof by solving linear recurrence:

$$e_n = p(1 + e_{n+1}) + q(1 + e_{n-1})$$

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Fair Case for $T = \infty$

Likewise,
E[#bets for $T = \infty$]
$\geq$ E[#bets for $T < \infty$]
$= n(T\text{-}n) \to \infty$ (as $T \to \infty$)

So the expected #bets to go broke is infinite!

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Team Problems

Problems 1—3

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