Quiz 2

Your name:____

Circle the name of your Tutorial Instructor:

Adrian Georgi Josh Karen Lee Min Nikos Tina

- This quiz is **closed book**. There is an **Appendix** with standard definitions.
- There are five (5) problems totaling 100 points. Total time is 120 minutes.
- Put your name on the top of **every** page – *these pages may be separated for grading*.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- You may assume any of the results presented in class or in the lecture notes.
- **Be neat and write legibly**. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it.
- GOOD LUCK!

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	15		
2	20		
3	20		
4	25		
5	20		
Total	100		

Problem 1 (15 points). Chuck Vest is planning to set aside $100M of MIT endowment funds in the year 2002 for student scholarships. His plan is to offer the same annual amount for student scholarships, starting as soon as the money is available in 2002 and continuing through 2102, at which point the $100M will be exhausted.

(a) (6 points) Assuming Chuck can count on earning a 3% annual return on the funds he sets aside, how much scholarship money can he offer each year? Let $r ::= 1/1.03$, and find a closed form formula in terms of r for the annual number of dollars; you do not have to evaluate the formula. (Full credit for a formula that evaluates to the right answer. Partial credit for an incorrect formula will only be considered when it is very clearly explained.)

(b) (9 points) Chuck would like to grant \$4M annually until 2102, but he realizes that the \$100M available is not enough to do this. So he decides to hold off on awarding scholarships for d years until the \$100M, earning 3% annually, has grown enough to award \$4M in annual scholarships for the remaining years until 2102. How many years must Chuck wait before he can start awarding these scholarships? Find a closed form formula in terms of r for the number of years, d, that Chuck must wait; you do not have to evaluate the formula. (Full credit for a formula that evaluates to the right answer. Partial credit for an incorrect formula will only be considered when it is very clearly explained.)

Problem 2 (20 points). Recall that for functions f, g on the natural numbers, $\mathbb{N}$, $f = O(g)$ iff

$$\exists c \in \mathbb{N} \, \exists n_0 \in \mathbb{N} \, \forall n \geq n_0 \quad c \cdot g(n) \geq |f(n)| \, . \tag{1}$$

For each pair of functions below, determine whether $f = O(g)$ and whether $g = O(f)$. In cases where one function is O() of the other, indicate the *smallest natural number*, c, and for that smallest c, the *smallest corresponding natural number* n_0 ensuring that condition (1) applies.

(a) (6 points) $f(n) = n^2, g(n) = 3n$.

$f = O(g)$ YES NO If YES, $c =$ ______, n_0 = ______

$g = O(f)$ YES NO If YES, $c =$ ______, n_0 = ______

(b) (7 points) $f(n) = (3n - 7)/(n + 4), g(n) = 4$

$f = O(g)$ YES NO If YES, $c =$ ______, n_0 = ______

$g = O(f)$ YES NO If YES, $c =$ ______, n_0 = ______

(c) (7 points) $f(n) = 1 + (n \sin(n\pi/2))^2, g(n) = 3n$

$f = O(g)$ YES NO If yes, $c =$ ______ n_0 = ______

$g = O(f)$ YES NO If yes, $c =$ ______ n_0 = ______

Problem 3 (20 points). Theory Hippo received a mysterious bag in the mail. When he opened the bag he found that it contained 100 6-sided dice and a note that said that 99 of the dice were fair, but one always throws a 6. Theory Hippo decides to experiment with the contents of the bag by pulling a die out of the bag at random, throwing it k times and then putting it back. Call this experiment $X(k)$.

(a) (5 points) Theory Hippo decides perform $X(10)$.

- Let a be the probability that Theory Hippo gets 10 sixes given that he pulled out a fair die. What is the value of a?

- Let b be the probability that Theory Hippo gets 10 sixes when he performs X(10). What is the value of b? You may (but are not required to) express b in terms of a above.

(b) (7 points) Theory Hippo has just performed $X(10)$ and ended up getting 10 sixes! Let c be the probability that he pulled out the unfair die, given that he got 10 sixes. What is the value of c? You may (but are not required to) express c in terms of a and/or b.

(c) (8 points) Theory Hippo's assistant performs $X(20)$ and reports that the *last* 10 of the 20 throws were sixes. What is the probability, f, that the *first* 10 throws were also sixes? You may (but are not required to) express f in terms of a, b, and/or c.

Problem 4 (25 points). We are going to classify different counting problems by figuring out which ones have the same formula. Here is a set of six formulas.

n^m:	____________
m^n:	____________
$P(n, m)$:	____________
$C(n-1+m, m)$:	____________
$C(n-1+m, n)$:	____________
2^{mn}:	____________

For each problem part below, write its label—(a),(b), . . . —on the line next to the corresponding formula above.

(a) (2 points) Number of arrangements of m indistinguishable balls in n distinguishable urns.

(b) (2 points) Number of arrangements of m distinguishable balls in n distinguishable urns.

(c) (2 points) Number of m letter words from an alphabet of size n, with no letter used more than once ($m \leq n$).

(d) (2 points) Number of m letter words from an alphabet of size n, where any letter can be repeated any number of times.

(e) (2 points) Number of sequences of n indistinguishable red balls and $m-1$ indistinguishable blue balls.

(f) (3 points) Number of matrices of size $\sqrt{n} \times \sqrt{n}$ with m possible values for each entry. (Assume n is a perfect square.)

(g) (3 points) Number of possible subsets of A, where $|A| = nm$.

(h) (3 points) Number of functions from set A to B where $|A| = n$ and $|B| = m$.

(i) (3 points) Number of relations from set A to B where $|A| = n$ and $|B| = m$.

(j) (3 points) Number of injections from set A to B where $|A| = m$ and $|B| = n$, and $m \leq n$.

Problem 5 (20 points). Let $S ::= \{0, 1, \ldots, 9\}^{90}$ be the set of all sequences of length 90 consisting of digits $\{0, 1, \ldots, 9\}$. Two sequences, $s_1, s_2 \in S$ are said to have the *same digit distribution* iff s_1 has the same of number of occurrences of digit j as s_2, for each $j \in \{0, \ldots, 9\}$.

(a) (7 points) Describe a bijection between digits distributions and arrangements of "stars & bars."

(b) (7 points) Give a closed formula, possibly also involving factorials and binomial coefficents, for the number, g, of possible digit distributions.

(c) (6 points) Let n be the smallest number such that any set of n sequences in S is sure to contain four different sequences with the same digit distribution. Give a closed formula, possibly also involving factorials and binomial coefficents, for n. You may write your formula in terms of g in part (b).