Solutions to In-Class Problems — Week 9, Wed

Problem 1. We have just demonstrated how to "guess" the 5th card in a poker hand when a collaborator reveals the other 4 cards. Describe a method for guessing 2 hidden cards in a hand of 9 cards when your collaborator reveals the other 7 cards.

Solution. Since there must be $\lceil 9/4 \rceil = 3$ cards with the same suit, our collaborator chooses to hide two of them and then use the third one as the first card to be revealed. So this first revealed card fixes the suit of the two hidden cards; it will also be used as the origin for the offset position of the first hidden card. This first hidden card will in turn be used as the origin for the offset of the other hidden card. There are six cards to code the two offset positions. These suffice to code two offsets of size from one to six. That is, our collaborator can choose one of the $3! = 6$ orders in which to reveal the first three cards and thereby tell us the offset position of the first hidden card. Our collaborator can then choose the order of the final three cards to describe the offset position of the second hidden card from the first. Note that the first revealed card must be chosen so that both offsets are less ≤ 6; since the sum of the offsets between successive cars ordered in a cycle from Ace to King is 13, it is not possible for more than one offset between successive cards to exceed seven, so this is always possible. ■

Problem 2.

(a) Prove the following corollary to Hall's Marriage Theorem.

Corollary 2.1. *Let $G = (V_1, V_2, E)$ be a bipartite graph where $|V_1| \leq |V_2|$. Suppose that a positive integer k exists such that each node in V_1 is adjacent to at least k vertices in V_2 and each node in V_2 is adjacent to at most k vertices in V_1. Then, G contains a perfect matching on the vertices in V_1.*

Solution. *Proof.* For any set, S of nodes, let $\mathrm{N}(S)$ be the nodes that are adjacent to some node in S, and let $\mathrm{E}(S)$ be the set of edges incident on some node in S. That is

$$\begin{aligned} \mathrm{N}(S) &::= \{u \mid \exists s \in S\ (u, s) \in E\} \\ \mathrm{E}(S) &::= \{(u, s) \mid u \in V_1 \cup V_2 \text{ and } s \in S\}. \end{aligned}$$

(Remember that the graph is undirected, so (u, v) and (v, u) describe the same edge.)

Consider any subset $S \subseteq V_1$. Since G is bipartite, $\text{N}(S) \subseteq V_2$. Since every edge incident to a node in S is by definition also incident to a node in $\text{N}(S)$, we have $\text{E}(S) \subseteq \text{E}(\text{N}(S))$ so

$$|\text{E}(S)| \leq |\text{E}(\text{N}(S))|.$$

Since each node in V_1 has at least k incident edges, we have

$$k \cdot |S| \leq |\text{E}(S)|.$$

Likewise, since each node in V_2 has at most k incident edges,

$$|\text{E}(\text{N}(S))| \leq k \cdot |\text{N}(S)|.$$

Combining these inequalities yields

$$k \cdot |S| \leq |\text{N}(S)| \leq |\text{E}(\text{N}(S))| \leq k \cdot |\text{N}(S)|.$$

Since $k > 0$, we conclude that

$$|S| \leq |\text{N}(S)|,$$

and by Hall's Marriage Theorem, the graph contains a perfect matching on the vertices in V_1. □

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(b) Use part (a) to determine the largest-size deck for which the trick of reading 4 cards from a 5-card hand remains possible. (You must not only determine an upper bound, you must prove the bound is achievable.)

Solution. If the trick is possible with a deck of n cards, then n must satisfy

$$\binom{n}{5} \leq P(n, 4)$$

where $n \geq 5$, since we must have at least 5 cards in the deck. That is,

$$\frac{n!}{(n-5)!5!} \leq n(n-1)(n-2)(n-3) = \frac{n!}{(n-4)!}.$$

Multiplying both sides by $(n-4)!5!/n!$ yields

$$n - 4 \leq 5!,$$

or $n \leq 5! + 4 = 124$. Consequently, any deck that allows the trick to work contains *at most* 124 cards.

To show that the trick indeed works with a 124-card deck, we use part (a).

Construct the bipartite graph $G = (V_1, V_2, E)$, where V_1 is the set of 5-card hands, V_2 is the set of sequences of 4 cards, and an edge in E connects a hand $u \in V_1$ to a sequence $v \in V_2$ if v can be obtained by removing a card from u and arranging the 4 remaining cards in sequence.

The degree of every node in V_1 is $P(5,4) = 120$, since there are $P(5,4)$ 4-card sequences from a set of 5 cards.

The degree of every node in V_2 is also 120, since there are exactly $120 = 124 - 4$ other cards in the 124-card deck that can be added to the sequence of 4 cards to make a 5-card hand.

Therefore, by part (a), a perfect matching exists. Thus, given any hand, there is a way of removing a card and arranging the remaining 4 cards in a sequence so that the sequence uniquely identifies the hand. This is precisely what's needed for the trick to work. ■