Solutions to In-Class Problems — Week 9, Mon

Problem 1. Theory Hippo recently lost a lot of money playing poker with the 6.042 staff. So, Theory Hippo has decided to spend a week in Las Vegas studying gambling. The hotel manager told Theory Hippo that a full house (three-of-a-kind and one pair) is much more valuable than two pairs. Since Theory Hippo is not so sure, he decides to count the number of hands with two pairs, but no three- or four-of-a-kind. In other words, the hand $\langle 9\clubsuit, 9\diamondsuit, 8\clubsuit, 8\heartsuit, 7\spadesuit\rangle$ is considered two pairs, but $\langle 9\clubsuit, 9\diamondsuit, 8\clubsuit, 8\heartsuit, 9\heartsuit\rangle$ and $\langle 9\clubsuit, 9\diamondsuit, 9\heartsuit, 9\spadesuit, 7\clubsuit\rangle$ are not.

Theory Hippo reasons that there are 13 choices for the value of the cards in the first pair. Then, there are $\binom{4}{2} = 6$ ways to choose 2 of the 4 cards in the deck that have this value. That leaves 12 choices for the value of the cards in the second pair, and $\binom{4}{2} = 6$ ways to choose 2 of the 4 cards with this value. Finally, the fifth card can be any one of the 44 cards remaining in the deck, because the fifth card can not have the same value as the first two pairs leaving $48 - 4$ possibilities. Altogether there are $13 \cdot 6 \cdot 12 \cdot 6 \cdot 44 = 247,104$ hands with just two pairs.

Is Theory Hippo's argument correct? If so, prove it by arguing that everything has been counted exactly once. If not, explain why and how to fix it.

Solution. (Also in the Course Notes.) The argument is incorrect. The bug is that every hand has been counted twice! For example, a pair of kings and a pair of queens is counted once with the kings as the first pair and a second time with the queens as the first pair. The references in the argument to a "first pair" and a "second pair" signal danger; these terms imply an ordering that is not part of the problem. There is a 2-to-1 mapping from the set of hands we counted to the set of hands with two pairs. Therefore, the set of hands with two pairs is half as large as our initial count:

$$\begin{aligned}\text{hands with two pairs} &= \frac{13 \cdot 6 \cdot 12 \cdot 6 \cdot 44}{2} \\ &= 123,552.\end{aligned}$$

The course notes show an alternate way of counting the hands with two-pairs, and one can verify by comparing the two answers. But, in any counting argument, one must keep track of whether order is important and make sure that things are counted only once.

A two-pairs hand turns out to be 33 times more likely than a full house. ■

Problem 2. One algorithm for checking whether an arbitrary labeled graph is 3-colorable is to enumerate all possible 3-color assignments for the vertices of the graph and then test each color assignment to see if it is a valid coloring. We are going to count the maximum number of color assignments that this algorithm may have to test. (In other words, what is the worst-case behavior of this approach?)

Let $G = (V, E)$ be an n-vertex graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, and let $C = \{R, G, B\}$ be the set of colors. A *color assignment* is a function $f : V \to C$. (*Note:* a *valid* coloring is one where $f(u) \neq f(v)$ for all $(u, v) \in E$, that is, no two adjacent vertices in the graph have the same color. We will not be concerned much with validity in this problem.)

(a) How many different color assignments are there? (Colors are not interchangable.)

Solution. This is exactly the number of functions from an n-element set (the vertices of G) to a 3-element set (the colors). We know there are 3^n such functions. ■

(b) There are much easier ways to check if a graph can be colored with 1 color or 2 colors. Therefore, we are really only interested in color assignments that use *exactly* 3 colors. How many ways can we color the vertices of G, so that all three colors in C are used?

Solution. Let A_i be the set of all colorings that use *exactly* i colors, for $i = 1, 2, 3$, so that we are after $|A_3|$. Clearly, A_1, A_2, A_3 are disjoint, so

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| ,$$

and they cover the entire set of 3-colorings of G, so

$$|A_1 \cup A_2 \cup A_3| = 3^n ,$$

by the previous part. Hence, to calculate $|A_3|$, we combine these two equations to get

$$|A_3| = 3^n - |A_2| - |A_1| , \tag{1}$$

so that it suffices to calculate $|A_2|$ and $|A_1|$.

The number $|A_1|$ of colorings that use exactly one color is easily seen to be

$$|A_1| = 3 , \tag{2}$$

because, once we pick one of the 3 colors, the numbering is fully specified.

To find the number $|A_2|$ of colorings that use exactly 2 colors, we first observe that

$$A_2 = A_{RG} \cup A_{GB} \cup A_{BR},$$

where A_{RG} is the set of colorings that use *both* red *and* green; and similarly for A_{GB} and A_{BR}. Since these three sets are disjoint and all have the same size, we have

$$|A_2| = |A_{RG}| + |A_{GB}| + |A_{BR}| = 3|A_{RG}| , \tag{3}$$

and we need to find $|A_{RG}|$ only.

Now, we know that the total number of colorings that use red and green (but maybe not both of them) is 2^n. Among these colorings only two fail to use both colors: the "all-red" coloring and the "all-green" one. Thus, we have

$$|A_{RG}| = 2^n - 2. \tag{4}$$

Combining Equations (**??**)–(**??**) yields

$$|A_3| = 3^n - 3(2^n - 2) - 3 = 3^n - 3 \cdot 2^n + 3\ .$$

■

(c) For a graph with 3 vertices we can see that the number of color assignments with exactly 3 colors are just the number of possible permutations of R, G, B, which is 3!. Verify that your formula in part (b) gives the right answer for a 3-vertex graph.

Solution. $3^3 - 3 \cdot 2^3 + 3 = 27 - 24 + 3 = 6 = 3!$.

The formula $3!\binom{n}{3}3^{n-3}$, which is likely to be suggested by several tables as the correct solution to part (b), returns the correct value for $n = 3$:

$$3!\binom{3}{3}3^{3-3} = 6 \cdot 1 \cdot 1 = 6\ ,$$

but it tragically fails for other values, e.g., $n = 4$:

$$3!\binom{4}{3}3^{4-3} = 6 \cdot 4 \cdot 3 = 72 \neq 36 = 3^4 - 3 \cdot 2^4 + 3\ .$$

Always sanity-check your formulas on special cases, but don't trust "proof by example." ■

(d) Even so, many of the assignments are redundant. For example,in a 4-vertex graph, the color assignment $RGBB$ is essentially the same as $RBGG$, because if one fails, the other does also. In general, any two colorings are equivalent if one assignment can be achieved from the other by just swapping color names. If we remove these redundant color assignments, how much can we improve our number from part (b)?

Solution. Consider the binary relation on A_3 that relates a coloring c to another coloring c' iff c' can be obtained from c via a permutation of the colors. One can easily check this is an equivalence relation, that breaks A_3 into equivalence classes of $3! = 6$ members each. (To see why 6 is the size of an equivalence class: if you pick any coloring c from the class and apply to it all $3! = 6$ possible permutations of the 3 colors, you are certain to derive all members of the class exactly once.)

We want to count the number of these equivalence classes. But, since the classes are disjoint, cover the entire A_3, and all have size 6, their number must be

$$\frac{|A_3|}{6} = \frac{3^n - 3 \cdot 2^n + 3}{6}\ .$$

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