Solutions to In-Class Problems — Week 9, Fri

Problem 1. Recall the definition of a multinomial coefficient:

$$\binom{n}{r_1, r_2, \ldots, r_k} ::= \frac{n!}{r_1!\, r_2! \cdots r_k!}. \tag{1}$$

An alternative definition is

$$\binom{n}{r_1, r_2, \ldots, r_k} ::= \binom{n}{r_1} \cdot \binom{n - r_1}{r_2} \binom{n - r_1 - r_2}{r_3} \cdots \binom{n - r_1 - r_2 - \cdots - r_{k-1}}{r_k} \tag{2}$$

Prove that the two definitions are equivalent using both a (i) combinatorial and (ii) an algebraic argument.

Solution. See Week 9 Notes, Theorem 7.2 ∎

Problem 2. Suppose that we have a generalized n-letter "MISSISSIPPI" word whose letters break down as follows: m occurrences of "M", i occurrences of "I", s of "S", and p of "P". For example, for the word "MISSISSIPPI" breaks down as $(n; m, i, s, p) = (11, 1, 4, 4, 2)$. For each of the following formulas, identify which correctly describe the number of different permutations of the generalized "MISSISSIPPI" word. For the correct formulas, justify your answer using a combinatorial (not an algebraic) argument.

(a) $\binom{n}{m, i, s, p}$.

Solution. True. A straightforward application of the theorem from class. ∎

(b) $\binom{n}{p, i, m, s}$

Solution. True. The order of the lower arguments of the multinomial coefficient is not important. ■

(c) $\binom{n}{m, i, s, n-m-i-s}$

Solution. True, because $m+i+s+p=n$ implies $p=n-m-i-s$. ■

(d) $\binom{n}{m}\binom{n}{i}\binom{n}{s}\binom{n}{p}$

Solution. False. ■

(e) $\binom{n}{m}\binom{m}{i}\binom{i}{s}\binom{s}{p}$

Solution. False. ■

Problem 3. The U.S. House of Representatives has 435 seats. (In this context, a "seat" is not a chair, but a right to belong to the legislative body. Thus, all seats are indistinguishable.)

(a) In how many ways can the seats be distributed among three political parties?

Solution. Each distribution can be represented by 435 stars for the seats, and two bars to separate the stars into three groups (Democrat, Republican, and Ross Perot's Reform Party). So the answer is

$$\binom{435+2}{2}.$$

■

(b) In how many ways can the seats be distributed among three political parties so that no party has a majority? (This implies that a coalition of any two parties *does* form a majority.)

Solution. No party can have 218 seats or more. We've counted the total number of ways of distributing seats among three parties in part (a), and thus we'll subtract the number of ways one party could have a majority. We can count the number of ways a party can be in the majority by giving it 218 seats and distributing the remaining 217 seats among the three parties. Since there are three parties, the answer is

$$\binom{437}{2} - 3\binom{219}{2}.$$

■

(c) How do your solutions to parts (a) and parts (b) change if the "seats" are actually distinguishable chairs (numbered 1 through 435)?

Solution. Part (a): 3^{435}.

Part (b):

$$3^{435} - 3\sum_{i=218}^{435}\binom{435}{i}2^{435-i}.$$

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