Solutions to In-Class Problems — Week 15, Mon

Problem 1. Prove that the Central Limit Theorem implies the Weak Law of Large Numbers. *Hint:* The only properties of $N(y)$ needed in the proof are that $\lim_{y\to\infty} N(y) = 1$ and $\lim_{y\to\infty} N(-y) = 0$.

Solution. Note first that $\mu_{S_n} = n\mu$, $\text{Var}\left[S_n\right] = n\sigma^2$, and so $\sigma_{S_n} = \sigma\sqrt{n}$. Now,

$$\begin{aligned}\left|\frac{S_n}{n} - \mu\right| > \epsilon \quad & \text{iff} \quad |S_n - n\mu| > n\epsilon \\ & \text{iff} \quad \left|\frac{S_n - n\mu}{\sigma_{S_n}}\right| > \frac{n\epsilon}{\sigma_{S_n}} \\ & \text{iff} \quad |S_n^*| > \frac{\sqrt{n}\epsilon}{\sigma}.\end{aligned}$$

But for any real number $\beta > 0$,

$$\frac{\sqrt{n}\epsilon}{\sigma} > \beta$$

will hold for all large n. Hence, for any $\beta > 0$ and all large n,

$$\Pr\left\{\left|\frac{S_n}{n} - \mu\right| > \epsilon\right\} = \Pr\left\{|S_n^*| > \frac{\sqrt{n}\epsilon}{\sigma}\right\} \le \Pr\left\{|S_n^*| > \beta\right\}. \tag{1}$$

So

$$\begin{aligned}\lim_{n\to\infty} \Pr\left\{\left|\frac{S_n}{n} - \mu\right| > \epsilon\right\} & \le \lim_{n\to\infty} \Pr\left\{|S_n^*| > \beta\right\} && \text{(by (1))} \\ & = \lim_{n\to\infty} \Pr\left\{S_n^* > \beta\right\} + \Pr\left\{S_n^* < -\beta\right\} \\ & = 1 - N(\beta) + N(-\beta), && \text{(by the Central Limit Thm (8))}\end{aligned}$$

for all real numbers $\beta > 0$. By choosing β large enough, we can ensure that $N(\beta)$ is arbitrarily close to 1 and $N(-\beta)$ is arbitrarily close to 0, so that final term above is arbitrarily close to 1-1+0 = 0. Hence,

$$\lim_{n\to\infty} \Pr\left\{\left|\frac{S_n}{n} - \mu\right| > \epsilon\right\} = 0,$$

which is the Weak Law of Large Numbers. ■

NOTE: We didn't get to the following problem in class.

Problem 2. To clarify the somewhat subtle difference between the Weak and Strong Laws of Large Numbers, we will construct an example of a sequence $X_1, X_2, \ldots$ of mutually independent random variables that satisfies the Weak Law of Large Numbers, but not the Strong Law. The distribution of X_i will have to depend on i, because otherwise both laws would be satisfied.[1]

In particular, let $X_1, X_2, \ldots$ be a sequence of mutually independent random variables such that $X_1 = 0$, and for each integer $i > 1$,

$$\Pr\{X_i = i\} = \frac{1}{2i \log i}, \quad \Pr\{X_i = -i\} = \frac{1}{2i \log i}, \quad \Pr\{X_i = 0\} = 1 - \frac{1}{i \log i}.$$

Note that $\mu = \mathrm{E}[X_i] = 0$ for all i.

(a) Show that $\mathrm{Var}[S_n] = \Theta(n^2/\log n)$. *Hint:* $n/\log n > i/\log i$ for $2 \le i \le n$.

Solution.

$$\begin{aligned}
\mathrm{Var}[S_n] &= \sum_{i=1}^{\infty} \mathrm{Var}[X_i] && \text{(independent variance additivity)} \\
&= \mathrm{Var}[X_1] + \sum_{i=2}^{n} \mathrm{E}\left[X_i^2\right] - \mathrm{E}^2[X_i] \\
&= 0 + \sum_{i=2}^{n} i^2 \left(\frac{1}{i \log i} - 0\right) \\
&= \sum_{i=2}^{n} \frac{i}{\log i}. \\
&= \Theta(n^2/\log n). && \text{(see below)} \qquad (2)
\end{aligned}$$

To justify (2), note that $x/\log x$ is increasing for $x > e$ since its derivative $(1/\log x)(1 - 1/\log x)$ is

[1] This problem is adapted from Grinstead & Snell, *Intro. to Probability*, Ch.8, exercise 16, pp314–315, where is credited to David Maslen.

positive. So $n/\log n \geq i/\log i$ for $2 \leq i \leq n$. Therefore,

$$\begin{aligned}
\frac{n^2}{\log n} &= \sum_{i=1}^{n} \frac{n}{\log n} \\
&\geq \sum_{i=2}^{n} \frac{i}{\log i} \\
&\geq \sum_{i=\lceil n/2 \rceil}^{n} \frac{i}{\log i} \\
&\geq \sum_{i=\lceil n/2 \rceil}^{n} \frac{n/2}{\log n} \\
&\geq \frac{n+1}{2} \frac{n/2}{\log n} \\
&\geq \frac{1}{4} \frac{n^2}{\log n}.
\end{aligned}$$

■

(b) Show that the sequence $X_1, X_2, \ldots$ satisfies the Weak Law of Large Numbers, *i.e.*, prove that for any $\epsilon > 0$

$$\lim_{n \to \infty} \Pr\left\{ \left| \frac{S_n}{n} \right| \geq \epsilon \right\} = 0.$$

Solution.

$$\begin{aligned}
\Pr\left\{ \left| \frac{S_n}{n} \right| \geq \epsilon \right\} &= \Pr\left\{ \left| \frac{S_n}{n} - 0 \right| \geq \epsilon \right\} \\
&\leq \text{Var}\left[\frac{S_n}{n} \right] \frac{1}{\epsilon^2} && \text{(Chebychev Bound)} \\
&= \frac{\text{Var}\left[S_n \right]}{n^2} \frac{1}{\epsilon^2} \\
&= \Theta\left(\frac{1}{\epsilon^2 \log n}\right), && \text{(by (2))}
\end{aligned}$$

which goes to zero as n goes to ∞. ■

We now show that the sequence $X_1, X_2, \ldots$ does not satisfy the Strong Law of Large Numbers.

(c) (The first Borel-Cantelli lemma.) Let $A_1, A_2, \ldots$ be any infinite sequence of mutually independent events such that

$$\sum_{i=1}^{\infty} \Pr\left\{ A_i \right\} = \infty. \tag{3}$$

Prove that

$$\Pr\{\text{infinitely many } A_i \text{ occur}\} = 1.$$

Hint: We know that the probability that no A_i with $n \geq i \geq r$ occurs is

$$\leq e^{-\operatorname{E}[T_{r,n}]} \tag{4}$$

where $T_{r,n} ::= \sum_{i=r}^{n} I_{A_i}$ is the number of events A_i with $n \geq i \geq r$ that occur. What happens as $n \to \infty$?

Solution. Let K_r be the event that no A_i with $i \geq r$ occurs. Also, let $K_{r,n}$ be the event that no A_i with $n \geq i \geq r$ occurs. Finally, let K be the event that only finitely many A_i's occur. We must prove that $\Pr\{K\} = 0$.

We begin by computing $\lim_{n\to\infty} e^{-\operatorname{E}[T_{r,n}]}$:

$$\begin{aligned}
\operatorname{E}[T_{r,n}] &= \sum_{i=r}^{n} \operatorname{E}[I_{A_i}] && \text{(linearity of expectation)} \\
&= \sum_{i=r}^{n} \Pr\{A_i\} && \text{(expectation of indicator variable)}
\end{aligned}$$

If we take the the limit as $n \to \infty$ we have:

$$\begin{aligned}
\lim_{n\to\infty} \operatorname{E}[T_{r,n}] &= \lim_{n\to\infty} \sum_{i=r}^{n} \Pr\{A_i\} \\
&= \infty && \text{(by (3))}.
\end{aligned}$$

Since $e^x \to 0$ as $x \to -\infty$, we conclude that:

$$\lim_{n\to\infty} e^{-\operatorname{E}[T_{r,n}]} = 0$$

Note that $K_r \subset K_{r,n}$ for any r, n. Hence $\Pr\{K_r\} \leq \Pr\{K_{r,n}\} \leq e^{-\operatorname{E}[T_{r,n}]}$. We have just proved that the right hand side tends to zero as n goes to ∞. Since the left hand side does not depend on n, and the inequality holds for all n s.t. $n \geq r$, we conclude that $\Pr\{K_r\}$ must be zero.

Now note that $K = \bigcup_r K_r$, so by Boole's law, $\Pr\{K\} \leq \sum_r \Pr\{K_r\}$, and since we've just proved that $\Pr\{K_r\} = 0$ for all r, it follows that $\Pr\{K\} = 0$. Hence the probability that infinitely many A_i's occur is 1. ■

(d) Show that $\sum_{i=1}^{\infty} \Pr\{|X_i| \geq i\}$ diverges. *Hint:* $\int 1/(xlogx)\,dx = \log\log x$.

Solution. $\Pr\{|X_i| \geq i\} = 1/(i \log i)$, so

$$\begin{aligned}\sum_{i=1}^{n} \Pr\{|X_i| \geq i\} &= 0 + \sum_{i=2}^{n} \frac{1}{i \log i} \\ &\geq \int_{2}^{n+1} \frac{1}{x \log x} dx \\ &= \log\log(n+1) - \log\log 2,\end{aligned}$$

and this last term approaches infinity and n approaches infinity. ■

(e) Conclude that

$$\Pr\left\{\lim_{n\to\infty} \frac{S_n}{n} = \mu\right\} = 0. \tag{5}$$

and hence that the Strong Law of Large Numbers *completely* fails for the sequence $X_1, X_2, \ldots$.

Hint:

$$\frac{X_n}{n} = \frac{S_n}{n} - \frac{n-1}{n}\frac{S_{n-1}}{n-1},$$

so if $\lim_{n\to\infty} S_n/n = 0$, then also $\lim_{n\to\infty} X_n/n = 0$.

Solution. By parts (c) and (d), the probability that $|X_i| \geq i$ for infinitely many i is 1. But if $|X_i| \geq i$ for infinitely many i, then by definition of the limit, $\lim_{n\to\infty} X_n/n \neq 0$. Hence,

$$\Pr\left\{\lim_{n\to\infty} X_n/n \neq 0\right\} = 1,$$

which means

$$\Pr\left\{\lim_{n\to\infty} X_n/n = 0\right\} = 0, \tag{6}$$

But the hint implies that

$$\Pr\left\{\lim_{n\to\infty} \frac{S_n}{n} = 0\right\} \leq \Pr\left\{\lim_{n\to\infty} \frac{X_n}{n} = 0\right\}. \tag{7}$$

Now (6) and (7) immediately imply (5). ■

A Appendix

The *probability density function (pdf)* for a random variable, R, is the function $f_R : \text{range}(R) \to [0,1]$ defined by:

$$f_R(x) ::= \Pr\{R = x\}.$$

Random variables $R_1, R_2, \ldots$ are *mutually independent* iff

$$\Pr\left\{\bigcap_i [R_i = x_i]\right\} = \prod_i \Pr\{R_i = x_i\},$$

for all $x_1, x_2, \dots \in \mathbb{R}$. They are *k-wise independent* iff $\{R_i \mid i \in J\}$ are mutually independent for all subsets $J \subset \mathbb{N}$ with $|J| = k$.

Theorem (Weak Law of Large Numbers). *Let $S_n ::= \sum_{i=1}^n X_i$, where $X_1, \ldots, X_n, \ldots$ are pairwise independent variables with the same expectation, μ, and standard deviation, σ. For any $\epsilon > 0$,*

$$\lim_{n\to\infty} \Pr\left\{\left|\frac{S_n}{n} - \mu\right| \geq \epsilon\right\} = 0.$$

Theorem (The Strong Law of Large Numbers). *Let $S_n ::= \sum_{i=1}^n X_i$ where $X_1, \ldots, X_i, \ldots$ are mutually independent, identically distributed random variables with finite expectation, μ. Then*

$$\Pr\left\{\lim_{n\to\infty} \frac{S_n}{n} = \mu\right\} = 1.$$

Definition. For any random variable, R, with finite mean, μ_R, and deviation, σ_R, let R^* be the random variable

$$R^* ::= \frac{R - \mu_R}{\sigma_R}.$$

R^* is called the "normalized" version of R.

Definition. The *normal density function* is the function

$$\eta(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

and the *normal distribution function* is its integral

$$N(y) = \int_{-\infty}^{y} \eta(x)dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2}dx.$$

The function $\eta(x)$ defines the standard *Bell curve*, centered about the origin with height $1/\sqrt{2\pi}$ and about two-thirds of its area within unit distance of the origin. The normal distribution function $N(y)$ approaches 0 as $y \to -\infty$. As y approaches zero from below, $N(y)$ grows rapidly towards $1/2$. Then as y continues to increase beyond zero, $N(y)$ rapidly approaches 1.

Theorem (Central Limit Theorem). *Let $S_n = \sum_{i=1}^n X_i$ where $X_1, \ldots, X_i, \ldots$ are mutually independent variables with the same mean, μ, and deviation, σ, and let S_n^* be the normalized version of S_n. Then*

$$\lim_{n\to\infty} \Pr\{S_n^* \leq \beta\} = N(\beta) \tag{8}$$

for any real number β.