Massachusetts Institute of Technology
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# Solutions to In-Class Problems — Week 13, Wed

**Problem 1.** Let $R$ be the number of heads that come up when we toss $n$ independent coins, where each coin comes up heads with probability $p$. The random variable $R$ has a binomial distribution. Using the formula for variance of sums, show that $\text{Var}[R] = np(1-p)$.

**Solution.** *From the course notes:* Recall that if a random variable, $R$, has a binomial distribution, then

$$\Pr\{R = k\} = \binom{n}{k} p^k (1-p)^{n-k}$$

where $n$ and $p$ are parameters such that $n \geq 1$ and $0 < p < 1$.

We can think of $R$ as the sum of $n$ independent Bernoulli variables. Formally, we can write $R = R_1 + R_2 + \cdots + R_n$ where

$$R_i = \begin{cases} 1 & \text{with probability } p, \\ 0 & \text{with probability } 1-p. \end{cases}$$

Now we can compute the variance of the binomially distributed variable $R$.

$$\begin{aligned}
\text{Var}[R] &= \text{Var}[R_1] + \text{Var}[R_2] + \ldots + \text{Var}[R_n] && \text{(Theorem 4.1)} \\
&= n \,\text{Var}[R_1] && (\text{Var}[R_i] = \text{Var}[R_j]) \\
&= n(\text{E}\left[R_1^2\right] - \text{E}^2[R_1]) && \text{(Def. of variance)} \\
&= n(\text{E}[R_1] - \text{E}^2[R_1]) && (R_1^2 = R_1) \\
&= n(p - p^2) = np(1-p). && (\text{E}[R_1] = \Pr\{R_1 = 1\} = p)
\end{aligned}$$

This shows that the binomial distribution has variance $np(1-p)$ and standard deviation $\sqrt{np(1-p)}$. In the special case of an unbiased binomial distribution ($p = 1/2$), the variance is $n/4$ and the standard deviation is $\sqrt{n}/2$. ■

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**Problem 2.** Provide an example that shows that the variance of the sum of two random variables is not necessarily equal to the sum of their variances, when the random variables are not independent.

**Solution.** A dramatic example is to take $Y = -X$. Then the sum of the two random variables is 0, so the variance of $X + Y$ is 0. But the sum of the variances is $2\,\text{Var}[X]$, since $Y$ has the same variance as $X$. Another example: let $X$ be equal to 1 if a flipped coin is heads, 0 otherwise. Similarly, let $Y$ be 1 if the same coin comes up tails. Then $\text{Var}[X] = \text{Var}[Y] = 1/4$, but $X + Y = 1$, so $\text{Var}[X + Y] = 0$. ■

**Problem 3.** The hat-check staff has had a long day, and at the end of the party they decide to return people's hats at random. Suppose that $n$ people have their hats returned at random. We previously showed that the expected number of people who get their own hat back is 1, irrespective of the total number of people. In this problem we will calculate the variance in the number of people who get their hat back.

Let $X_i = 1$ if the $i$th person gets his or her own hat back and 0 otherwise. Let $S_n = \sum_{i=1}^{n} X_i$, so $S_n$ is the total number of people who get their own hat back. Show that

**(a)** $\text{E}\left[X_i^2\right] = 1/n$.

**Solution.** $X_i = 1$ with probability $1/n$ and 0 otherwise. Thus $X_i^2 = 1$ with probability $1/n$ and 0 otherwise. So $\text{E}\left[X_i^2\right] = 1/n$. ■

**(b)** $\text{E}[X_i X_j] = 1/n(n-1)$ for $i \neq j$.

**Solution.** The probability that $X_i$ and $X_j$ are both 1 is $1/n \cdot 1/(n-1) = 1/n(n-1)$. Thus $X_i X_j = 1$ with probability $1/n(n-1)$, and is zero otherwise. So $\text{E}[X_i X_j] = 1/n(n-1)$.

■

**(c)** $\text{E}\left[S_n^2\right] = 2$. *Hint:* Use (a) and (b).

**Solution.**

$$\begin{aligned}
\text{E}\left[S_n^2\right] &= \sum_i \text{E}\left[X_i^2\right] + \sum_i \sum_{j \neq i} \text{E}[X_i X_j] \\
&= n \cdot \frac{1}{n} + n(n-1) \cdot \frac{1}{n(n-1)} \\
&= 2.
\end{aligned}$$

■

**(d)** $\text{Var}[S_n] = 1$.

**Solution.**

$$\begin{aligned}\text{Var}[S_n] &= \text{E}\left[S_n^2\right] - \text{E}^2[S_n]\\ &= 2 - (n(1/n))^2\\ &= 2 - 1\\ &= 1.\end{aligned}$$

■

**(e)** Explain why you can not use the variance of sums formula to calculate $\text{Var}[S_n]$.

**Solution.** The indicator random variables, $X_i$, are not even pairwise independent. This can be seen by comparing the marginal and conditional probability of a particular person, Alice, getting her hat back. The marginal probability, unconditioned on any other events, is $1/n$ as we've computed before. However, if compute this probability conditioned on the event that a second person, Bob, got his hat back, we find that the probability of Alice getting her hat back is $1/(n-1)$. ■

**(f)** Using Chebyshev's Inequality, show that $\Pr\{S_n \geq 11\} \leq .01$ for any $n \geq 11$.

**Solution.**

$$\begin{aligned}\Pr\{S_n \geq 11\} &= \Pr\{S_n - \text{E}[S_n] \geq 11 - \text{E}[S_n]\}\\ &= \Pr\{S_n - \text{E}[S_n] \geq 10\}\\ &\leq \frac{\text{Var}[S_n]}{10^2} = .01\end{aligned}$$

Note that the $X_i$'s are Bernoulli variables but are *not* independent, so $S_n$ does not have a binomial distribution and the binomial estimates from Lecture Notes do not apply. ■

**Problem 4.** Prove that $\text{Var}[X+Y+Z] = \text{Var}[X] + \text{Var}[Y] + \text{Var}[Z]$, if $X, Y, Z$ are pairwise independent. Explain why pairwise independence is sufficient.

*Reminder:* If two random variable are independent, then $\text{E}[X_iX_j] = \text{E}[X_i]\,\text{E}[X_j]$. Given a set of random variables, pairwise independence implies that every possible pair of random variables is independent.

**Solution.** Let $R = X + Y + Z$. We can compute $\text{E}^2[R]$ as follows:

$$\begin{aligned}\text{E}^2[R] &= (\text{E}[X+Y+Z])^2\\ &= (\text{E}[X] + \text{E}[Y] + \text{E}[Z])^2\\ &= \text{E}^2[X] + \text{E}^2[Y] + \text{E}^2[Z] + 2\,\text{E}[X]\,\text{E}[Y] + 2\,\text{E}[X]\,\text{E}[Z] + 2\,\text{E}[Y]\,\text{E}[Z]\end{aligned}$$

Computing $\mathrm{E}\left[R^2\right]$ we get:

$$\begin{aligned}
\mathrm{E}\left[R^2\right] &= \mathrm{E}\left[(X+Y+Z)^2\right] \\
&= \mathrm{E}\left[X^2+Y^2+Z^2+2XY+2XZ+2YZ\right] \\
&= \mathrm{E}\left[X^2\right]+\mathrm{E}\left[Y^2\right]+\mathrm{E}\left[Z^2\right]+2\,\mathrm{E}\left[XY\right]+2\,\mathrm{E}\left[XZ\right]+2\,\mathrm{E}\left[YZ\right] \\
&= \mathrm{E}\left[X^2\right]+\mathrm{E}\left[Y^2\right]+\mathrm{E}\left[Z^2\right]+2\,\mathrm{E}\left[X\right]\mathrm{E}\left[Y\right]+2\,\mathrm{E}\left[X\right]\mathrm{E}\left[Z\right]+2\,\mathrm{E}\left[Y\right]\mathrm{E}\left[Z\right]
\end{aligned}$$

Notice that the last step is only valid because the random variables are pairwise independent. Finally, we can compute $\mathrm{Var}\left[R\right]$. We can begin immediately by cancelling out the cross terms leaving us with:

$$\begin{aligned}
\mathrm{Var}\left[R\right] &= \mathrm{E}\left[R^2\right]-\mathrm{E}^2\left[R\right] \\
&= \mathrm{E}\left[X^2\right]+\mathrm{E}\left[Y^2\right]+\mathrm{E}\left[Z^2\right]-(\mathrm{E}^2\left[X\right]+\mathrm{E}^2\left[Y\right]+\mathrm{E}^2\left[Z\right]) \\
&= \mathrm{E}\left[X^2\right]-\mathrm{E}^2\left[X\right]+\mathrm{E}\left[Y^2\right]-\mathrm{E}^2\left[Y\right]+\mathrm{E}\left[Z^2\right]-\mathrm{E}^2\left[Z\right] \\
&= \mathrm{Var}\left[X\right]+\mathrm{Var}\left[Y\right]+\mathrm{Var}\left[Z\right],
\end{aligned}$$

thus concluding the proof. ■

# A Appendix

Random variables $R_1, R_2, \ldots$ are *mutually independent* iff

$$\Pr\left\{\bigcap_i [R_i = x_i]\right\} = \prod_i \Pr\left\{R_i = x_i\right\},$$

for all $x_1, x_2, \cdots \in \mathbb{R}$. They are *k-wise independent* iff $\{R_i \mid i \in J\}$ are mutually independent for all subsets $J \subset \mathbb{N}$ with $|J| = k$.

**Theorem (Expectation of a Product).** *If $R_1, R_2, \ldots, R_n$ are mutually independent, then*

$$\mathrm{E}\left[R_1 \cdot R_2 \cdot \cdots \cdot R_n\right] = \mathrm{E}\left[R_1\right] \cdot \mathrm{E}\left[R_2\right] \cdots \mathrm{E}\left[R_n\right].$$

The *variance*, $\mathrm{Var}\left[R\right]$, of a random variable, $R$, is:

$$\mathrm{Var}\left[R\right] ::= \mathrm{E}\left[(R-\mathrm{E}\left[R\right])^2\right].$$

Variance can also be equivalently defined as:

$$\mathrm{Var}\left[R\right] ::= \mathrm{E}\left[R^2\right]-\mathrm{E}^2\left[R\right],$$

**Lemma.** *For $a, b \in \mathbb{R}$,*

$$\mathrm{Var}\left[aR+b\right] = a^2\,\mathrm{Var}\left[R\right]$$

**Theorem 4.1.** *If* $R_1, R_2, \ldots, R_n$ *are* pairwise *independent random variables, then*

$$\text{Var}\left[R_1 + R_2 + \cdots + R_n\right] = \text{Var}\left[R_1\right] + \text{Var}\left[R_2\right] + \cdots + \text{Var}\left[R_n\right].$$

**Theorem (Markov's Theorem).** *If R is a nonnegative random variable, then for all* $x > 0$

$$\Pr\left\{R \geq x\right\} \leq \frac{\text{E}\left[R\right]}{x}.$$

An alternative formulation is

$$\Pr\left\{R \geq x\,\text{E}\left[R\right]\right\} \leq \frac{1}{x}.$$

**Theorem (Chebyshev).** *Let* $R$ *be a random variable, and let* $x$ *be a positive real number. Then*

$$\Pr\left\{\left|R - \text{E}\left[R\right]\right| \geq x\right\} \leq \frac{\text{Var}\left[R\right]}{x^2}.$$

An alternative formulation is

$$\Pr\left\{\left|R - \text{E}\left[R\right]\right| \geq x\sigma_R\right\} \leq \frac{1}{x^2},$$

where $\sigma_R ::= \sqrt{\text{Var}\left[R\right]}$ is the standard deviation of $R$.