Solutions to In-Class Problems — Week 12, Mon

Problem 1. In the original game of Carnival Dice, the player chooses a number from 1 to 6. She then throws three fair and mutually independent dice. She wins one dollar if any die matches, and loses a dollar otherwise. This is a losing proposition for the player.

Consider a modified version of Carnival Dice. The game is the same except the player wins one dollar for *each* die that matches her number, and she loses one dollar if no die matches. Is this a good game to play? What is her expected profit?

Solution. At first glance the new game appears to be fair; after all, the player is now "justly compensated" if she rolls her number on more than one die. But its not much better.

Let the random variable R be the amount of money won or lost by the player in a round. We can compute the expected value of R as follows:

$$\begin{aligned}
\mathrm{E}[R] &= -1 \cdot \Pr\{0 \text{ matches}\} + 1 \cdot \Pr\{1 \text{ match}\} + 2 \cdot \Pr\{2 \text{ matches}\} + 3 \cdot \Pr\{3 \text{ matches}\} \\
&= -1 \cdot \left(\frac{5}{6}\right)^3 + 1 \cdot 3\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2 + 2 \cdot 3\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right) + 3 \cdot \left(\frac{1}{6}\right)^3 \\
&= \frac{-125 + 75 + 30 + 3}{216} \\
&= \frac{-17}{216}
\end{aligned}$$

Even with a \$3 payoff for three matching dice, the player can expect to lose $17/216$ of a dollar, or about 8 cents, in every round. This is still a horrible game for the player! ■

Problem 2. There is a dinner party where N people check their hats. The hats are mixed up during dinner, so that afterward each person receives a random hat. In particular, each person gets their own hat with probability $1/N$. What is the expected number of people who get their own hat?

Solution. Also from the Course Notes.

Let the random variable R be the number of people that get their own hat. The trick is to express R as a sum of indicator variables. In particular, let R_i be an indicator for the event that the ith person gets their own hat. That is, $R_i = 1$ is the event that they gets their own hat, and $R_i = 0$ is the event that they gets the wrong hat. The number of people that get their own hat is the sum of these indicators:

$$R = R_1 + R_2 + \cdots + R_N.$$

These indicator variables are *not* mutually independent. For example, if $N-1$ people all get their own hats, then the last person is certain to receive their own hat. But linearity of expectation, doesn't require indicator variables are independent, because no matter what, we can take the expected value of both sides of the equation above and apply linearity of expectation:

$$\mathrm{E}[R] = \mathrm{E}[R_1 + R_2 + \cdots + R_N] = \mathrm{E}[R_1] + \mathrm{E}[R_2] + \cdots + \mathrm{E}[R_N].$$

The expected value of an indicator variable is always the probability that the indicator is 1. In this case, the quantity $\Pr\{R_i = 1\}$ is the probability that the ith person gets his own hat, which is just $1/N$. We can now compute the expected number of people that get their own hat:

$$\begin{aligned} \mathrm{E}[R] &= \mathrm{E}[R_1] + \mathrm{E}[R_2] + \cdots + \mathrm{E}[R_N] \\ &= \frac{1}{N} + \frac{1}{N} + \cdots + \frac{1}{N} = 1. \end{aligned}$$

We should expect exactly one person to get the right hat! ■

Problem 3. Prove that $\mathrm{E}[R_1 + R_2] = \mathrm{E}[R_1] + \mathrm{E}[R_2]$, where R_1 and R_2 are two random variables defined on the same sample space, but not necessarily independent. (*Hint:* Start from Definition 4.1).

Solution. From the Course Notes:

Proof. Let $T ::= R_1 + R_2$. The proof follows straightforwardly by rearranging terms using Definition 4.1 of $\mathrm{E}[T]$.

$$\begin{aligned} \mathrm{E}[R_1 + R_2] &::= \mathrm{E}[T] \\ &::= \sum_{s \in \mathcal{S}} T(s) \cdot \Pr\{s\} && \text{(Def. 4.1)} \\ &= \sum_{s \in \mathcal{S}} (R_1(s) + R_2(s)) \cdot \Pr\{s\} && \text{(Def. of } T\text{)} \\ &= \sum_{s \in \mathcal{S}} R_1(s) \Pr\{s\} + \sum_{s \in \mathcal{S}} R_2(s) \Pr\{s\} && \text{(rearanging terms)} \\ &= \mathrm{E}[R_1] + \mathrm{E}[R_2]. && \text{(Def. 4.1)} \end{aligned}$$

□

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Problem 4. Let R be the number of heads that come up when we toss n independent coins, where each coin comes up heads with probability p. The random variable R has a binomial distribution. Prove that the expected value of R is np.

Solution. Let $H_{n,p}$ be the number of heads after the flips. Then $H_{n,p}$ has the binomial distribution with parameters n and p. Now let I_k be the indicator for the kth coin coming up heads. By Lemma 4.2, we have

$$\mathrm{E}\left[I_k\right] = p.$$

But

$$H_{n,p} = \sum_{k=1}^{n} I_k,$$

so by linearity

$$\mathrm{E}\left[H_{n,p}\right] = \mathrm{E}\left[\sum_{k=1}^{n} I_k\right] = \sum_{k=1}^{n} \mathrm{E}\left[I_k\right] = \sum_{k=1}^{n} p = pn.$$

That is, the expectation of a n, p-binomially distributed variable is pn. ■

A Appendix

Definition 4.1. The *expectation*, $\text{E}[R]$, of a random variable, R, on the sample space S, is defined as:

$$\text{E}[R] ::= \sum_{s \in \mathcal{S}} R(s) \cdot \Pr\{s\}. \tag{1}$$

Another equivalent definition is:

Definition 4.2. The *expectation* of random variable, R, is:

$$\text{E}[R] ::= \sum_{r \in \text{range}(R)} r \cdot \Pr\{R = r\}. \tag{2}$$

Theorem 4.3. *If R is a random variable with range $\mathbb{N}$, then*

$$\text{E}[R] = \sum_{i=0}^{\infty} \Pr\{R > i\}.$$

Theorem 4.4. *(Expectation of a Sum) For any random variables $R_1, \ldots, R_k$,*

$$\text{E}\left[\sum_{i=1}^{k} R_i\right] = \sum_{i=1}^{k} \text{E}[R_i].$$

Theorem 4.5. *If $|x| < 1$, then*

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}.$$