

## In-Class Problems — Week 12, Mon

**Problem 1.** In the original game of Carnival Dice, the player chooses a number from 1 to 6. She then throws three fair and mutually independent dice. She wins one dollar if any die matches, and loses a dollar otherwise. This is a losing proposition for the player.

Consider a modified version of Carnival Dice. The game is the same except the player wins one dollar for *each* die that matches her number, and she loses one dollar if no die matches. Is this a good game to play? What is her expected profit?

**Problem 2.** There is a dinner party where  $N$  people check their hats. The hats are mixed up during dinner, so that afterward each person receives a random hat. In particular, each person gets their own hat with probability  $1/N$ . What is the expected number of people who get their own hat?

**Problem 3.** Prove that  $E[R_1 + R_2] = E[R_1] + E[R_2]$ , where  $R_1$  and  $R_2$  are two random variables defined on the same sample space, but not necessarily independent. (*Hint:* Start from Definition 4.1).

**Problem 4.** Let  $R$  be the number of heads that come up when we toss  $n$  independent coins, where each coin comes up heads with probability  $p$ . The random variable  $R$  has a binomial distribution. Prove that the expected value of  $R$  is  $np$ .

## A Appendix

**Definition 4.1.** The *expectation*,  $E[R]$ , of a random variable,  $R$ , on the sample space  $S$ , is defined as:

$$E[R] ::= \sum_{s \in S} R(s) \cdot \Pr\{s\}. \quad (1)$$

Another equivalent definition is:

**Definition 4.2.** The *expectation* of random variable,  $R$ , is:

$$E[R] ::= \sum_{r \in \text{range}(R)} r \cdot \Pr\{R = r\}. \quad (2)$$

**Theorem 4.3.** If  $R$  is a random variable with range  $\mathbb{N}$ , then

$$E[R] = \sum_{i=0}^{\infty} \Pr\{R > i\}.$$

**Theorem 4.4.** (*Expectation of a Sum*) For any random variables  $R_1, \dots, R_k$ ,

$$E\left[\sum_{i=1}^k R_i\right] = \sum_{i=1}^k E[R_i].$$

**Theorem 4.5.** If  $|x| < 1$ , then

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}.$$