Solutions to In-Class Problems — Week 12, Fri

Problem 1. Consider the excerpt from Herbert Simon's paper "The Architecture of Complexity" attached at the end.

(a) What is wrong with the analysis?

(b) What is the correct expected time for Hora and Tempus to complete a watch?

Solution. The expected time for Tempus to complete an assembly is (expected length of a try) x (expected number of tries). Success is if he assembles all n parts correctly, which is probability $(1-p)^n$. By mean time to failure, the expected number of rounds is $1/(1-p)^n$. The expected length of a try (how far he expected to get before a piece breaks, is again a mean time to failure type argument but over a finite range. Turns out that E[length of a try] = $1-(1-p)^n/p$.

So the expected number of steps E[T] is $1/p((1/(1-p)^n)-1)$.

If p = .01 and $n = 1000$ this is roughly 2 million steps.

Hora on the other hand uses a heirarchy of subassemblies, where each subassembly has 10 elements. The expected time E[T] for $n = 10$ is about 10.57, same formula as above. But now if you draw out the tree with 1000 leaves, then there are 111 subassemblies (nodes) that need to get created. So total time = 111 x 10.57 = 1173.

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Problem 2. The St. Petersburg Casino offers the following game: the gambler bets a fixed wager, and then the dealer flips a fair coin (dealers do not flip coins in US casinos, but they do in St. Petersburg) until it comes up heads. The gambler receives \$1 if the coin shows heads the first time, \$2 if it shows the first head at second toss, and in general \$ 2^{k-1} if the dealer tosses the coin k times to get the first head.

(a) Suppose the fixed wager is \$10. What is the expected amount of money that the gambler will win in this game? Suppose the fixed wager is \$10,000?

Solution. Let V be the random variable corresponding to the amount of money that the gambler is paid by the dealer. The distribution of V is as follows: for any $n \geq 1$,

$$\Pr\left\{V = 2^{n-1}\right\} = 2^{-n},$$

where the probability refers to the event that the dealer tosses $n-1$ tails followed by a head. The average of V is

$$\mathrm{E}[V] = \sum_{n=1}^{\infty}[2^{-n}2^{n-1}] = \sum_{n=1}^{\infty} 1/2 = \infty.$$

So whatever the fixed wager, the gambler expects to win an infinite amount. ■

(b) What is the probability that the gambler does not lose money in a game when the fixed wager is \$10,000?

Solution. The probability of recovering the money of the wager is only

$$\begin{aligned}
\Pr\{V \geq 10,000\} &= \Pr\left\{V \geq 2^{14}\right\} \\
&= \Pr\{\text{dealer flips at least 14 consecutive tails}\} \\
&= 2^{-14} = \frac{1}{16,384}
\end{aligned}$$

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(c) In reality, it would not be reasonable for the gambler to play the game with the fixed wager at \$10,000. Why? (*Hint:* Suppose the casino has a limit of a billion dollars.)

Solution. Above, we concluded that a \$10,000 wager still "makes sense" because the expected amount of money (payoff) that the gambler will get from the casino is infinite. However, in reality, the casino can only pay back a certain amount of money before it goes bankrupt. This restriction has a dramatic impact on the expected value of the payoff. Consider, for example, the following, more realistic setting:

> The gambler still receives \$ 2^{k-1} if he flips $k-1$ tails before flipping a head, but this only holds for profits up to \$ 2^{30} (i.e. about a billion dollars). If the gambler flips 31 tails or more before he gets a head, he still only gets \$ 2^{30}.

Notice that the probability of flipping 31 or more tails in a row is 2^{-31}.

Under this scenario, the expected amount of money that the gambler will receive from the casino is

$$\begin{aligned}
\mathrm{E}[V] &= \sum_{n=1}^{31}\frac{1}{2} + 2^{-31}\cdot 2^{30} \\
&= 31\cdot\frac{1}{2} + \frac{1}{2} \\
&= 16
\end{aligned}$$

So, in our "realistic" scenario, the expected payoff is just $ 16. which is considerably less than infinity. The $ 10 wager makes sense, but not the $ 10,000 one.

You should note that the infinite expectation we got in part (a) was due to the (infinite) contribution of ridiculously high payoffs which, even though they happen with inversely ridiculously low probability, still contribute $\frac{1}{2}$ *each* to the sum. Since there is an infinite number of (different) ridiculously high payoffs, we have an infinite number of $\frac{1}{2}$ terms in the sequence, which makes the sum infinite.

Optional exercise: Above, by setting the maximum payoff at $\$2^{30}$, we brought the expected payoff down to just $ 16. What should the maximum payoff be for the expected payoff to be greater than $ 10,000, so that the $ 10,000 wager makes sense?

Answer: Around $\$\ 2^{20,000}$. For comparison, note that there are 2^{300} particles in the universe. ■

Problem 3. Just like event probabilities, expectations can be conditioned on some event. We define *conditional expectation*, $\mathrm{E}[R \mid A]$, *of a random variable*, R, *given event*, A:

$$\mathrm{E}[R \mid A] ::= \sum_r r \cdot \Pr\{R = r \mid A\}. \tag{1}$$

In other words, it is the expected value of the variable R once we skew the distribution of R to be conditioned on event A. A real benefit of conditional expectation is the way it lets us divide complicated expectation calculations into simpler cases.

Theorem 3.1. *[Law of Total Expectation] If the sample space is the disjoint union of events* $A_1, A_2, \cdots$, *then*

$$\mathrm{E}[R] = \sum_i \mathrm{E}[R \mid A_i] \Pr\{A_i\}.$$

Prove the above theorem.

Solution. *Proof.*

$$\begin{aligned}
\mathrm{E}[R] &::= \sum_r r \cdot \Pr\{R = r\} \\
&= \sum_r r \cdot \sum_i \Pr\{R = r \mid A_i\} \Pr\{A_i\} && \text{(Total Probability)} \\
&= \sum_r \sum_i r \cdot \Pr\{R = r \mid A_i\} \Pr\{A_i\} && \text{(distribute constant } r\text{)} \\
&= \sum_i \sum_r r \cdot \Pr\{R = r \mid A_i\} \Pr\{A_i\} && \text{(exchange order of summation)} \\
&= \sum_i \Pr\{A_i\} \sum_r r \cdot \Pr\{R = r \mid A_i\} && \text{(factor constant } \Pr\{A_i\}\text{)} \\
&= \sum_i \Pr\{A_i\} \mathrm{E}[R \mid A_i] && \text{(Def. 1).}
\end{aligned}$$

□

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Problem 4. Compute the expected value for each of the following random variables. (Assume that all dice are fair and six-sided and that dice rolls are mutually independent.)

(a) The sum of the rolls of three dice.

Solution. The expected value of one die roll is

$$1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} \quad = \quad \frac{7}{2}.$$

By linearity of expectation, the sum three rolls is

$$\frac{7}{2} + \frac{7}{2} + \frac{7}{2} \quad = \quad \frac{21}{2}.$$

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(b) The product of the rolls of three dice.

Solution. For independent random variables, the expectation of the product is the product of the expectations. Therefore, the expected product is

$$\frac{7}{2} \cdot \frac{7}{2} \cdot \frac{7}{2} \quad = \quad \frac{343}{8}.$$

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(c) The sum of the rolls of a number of dice that is given by the roll of a single die. (For example, if you roll a 3 on the single die, then you take the sum of 3 dice rolls; if you roll a 5 on the single die, then you take the sum of 5 dice rolls.)

Solution. By Wald's Theorem, the answer is the expected number of dice rolled multiplied by the expected roll of one die, which is

$$\frac{7}{2} \cdot \frac{7}{2} \quad = \quad \frac{49}{4}.$$

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(d) Suppose now that the dice rolls are not guaranteed to be mutually independent. Which of your answers above must still be correct?

Solution. The first and third answers remain valid because linearity of expectation and Wald's Theorem do not require independence. The remaining answer about the product of the dice is does make use of independence. ■

A Appendix

The *expectation* of random variable, R, is:

$$\mathrm{E}[R] ::= \sum_{r \in \text{range}(R)} r \cdot \Pr\{R = r\}.$$

If R has codomain $\mathbb{N}$, then this definition can also be written as

$$\mathrm{E}[R] = \sum_{r \in \mathbb{N}} \Pr\{R > r\}$$

Theorem. *Let $C_1, C_2, \ldots,$ be a sequence of nonnegative random variables, and let Q be a positive integer-valued random variable, all with finite expectations. Suppose that*

$$\mathrm{E}[C_i \mid Q \geq i] = \mu$$

for some $\mu \in \mathbb{R}$ and for all $i \geq 1$. Then

$$\mathrm{E}[C_1 + C_2 + \cdots + C_Q] = \mu\, \mathrm{E}[Q].$$

Architecture of Complexity
Herbert A. Simon
Professor of Administration, Carnegie Institute of Technology
Proceedings of the American Philosophical Society, vol 106, no 6, dec 1962. Excerpt, page 470.

THE EVOLUTION OF COMPLEX SYSTEMS

Let me introduce the topic of evolution with a parable. There were once two watch makers, named Hora and Tempus, who manufactured very fine watches. Both of then were highly regarded, and the phones in their workshops rang frequently — new costumers were constantly calling them. However, Hora prospered, while Tempus became poorer and poorer and finally lost his shop. What was the reason?

The watches the men made consisted of about 1,000 parts each. Tempus had so constructed his that if he had one partly assembled and had to put it down — to answer the phone say — it immediately fell to pieces and had to be reassembled from the elements. The better the customers liked his watches, the more they phoned him, the more difficult it became for him to find enough uninterrupted time to finish a watch.

The watches that Hora made were no less complex than those of Tempus. But he had designed them so that he could put together subassemblies of about ten elements each. Ten of these subassemblies, again, could be put together into a larger subassembly; and a system of ten of the latter subassemblies constituted the whole watch. Hence, when Hora had to put down a partly assembled watch in order to answer the phone, he lost only a small part of his work, and he assembled his watches in only a fraction of the man-hours it took Tempus.

It is rather easy to make a quantitative analysis of the relative difficulty of the tasks of Tempus and Hora: Suppose the probability that an interruption will occur while a part is being added to the incomplete assembly is p. Then the probability that Tempus can complete a watch he has started without interruption is $(1-p)^{1000}$ — a very small number unless p is .001 or less. Each interruption will cost, on average, the time to assemble $1/p$ parts (the expected number assembled before interruption). On the other hand, Hora has to complete one hundred eleven sub-assemblies of ten parts each. The probability that he will not be interrupted while completing any one of these is $(1-p)^{10}$, and each interruption will cost only about the time required to assemble five parts.

Now if p is about .01 — that is, there is one chance in a hundred that either watchmaker will be interrupted while adding any one part to an assembly — then a straightforward calculation shows that it will take Tempus, on the average, about four thousand times as long to assemble a watch as Hora. We arrive at the estimate as follows:

1. Hora must make 111 times as many complete assemblies as Tempus; but,
2. Tempus will lose on average 20 times as much work for each interrupted assembly as Hora [100 parts, on the average, as against 5]; and,
3. Tempus will complete an assembly only 44 times per million attempts ($.99^{1000} = 44 \times 10^{-6}$), while Hora will complete nine out of ten ($.99^{10} = 9 \times 10^{-1}$). Hence Tempus will have to make 20,000 as many attempts per completed assembly as Hora. $(9 \times 10^{-1})/(44 \times 10^{-6}) = 2 \times 10^4$. Multiplying these three ratios we get:
$$\frac{1}{111} \times \frac{100}{5} \times \frac{.99^{10}}{.99^{1000}} = \frac{1}{111} \times 20 \times 20,000 \sim 4,000.$$