Solutions to In-Class Problems — Week 11, Fri

Problem 1. Independently flip three fair coins, and define C to be the number of heads which appear and M to be 1 iff all three coins match and 0 otherwise. Also, let $S ::= C \bmod 2$.

What are the values of the probability density functions (pdf's) f_C, f_M, and f_S? Likewise for the distribution functions F_C, F_M, and F_S?

Solution.

$$\begin{aligned}
f_C(0) =& f_C(3) = 1/8 \\
f_C(1) =& f_C(2) = 3/8 \\
f_M(0) =& 3/4 \\
f_M(1) =& 1/4 \\
f_S(0) =& f_S(1) = 1/2.
\end{aligned}$$

$$F_C(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1/8 & \text{for } 0 \leq x < 1, \\ 1/2 & \text{for } 1 \leq x < 2, \\ 7/8 & \text{for } 2 \leq x < 3, \\ 1 & \text{for } 3 \leq x. \end{cases}$$

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Problem 2. Consider the binary relation ρ on the set of real-valued random variables,

$$X \rho Y \longleftrightarrow \Pr\{X \neq Y\} = 0.$$

(a) Give an example of RV's X, Y on some probability space, $\mathcal{S}$, such that $X \rho Y$ but $X \neq Y$.

Solution. Let

$$\begin{aligned} \mathcal{S} &::= \{\texttt{win, lose}\}, \\ \Pr\{\texttt{win}\} &::= 1, \\ \Pr\{\texttt{lose}\} &::= 0, \\ X(\texttt{win}) = Y(\texttt{win}) &::= 5, \\ X(\texttt{lose}) &::= 5, \\ Y(\texttt{lose}) &::= 4.99. \end{aligned}$$

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(b) Prove that ρ is an equivalence relation.

Solution. The trick is to observe that

$$\begin{aligned} \Pr\{X \neq Y\} = 0 &\longleftrightarrow \Pr\{X = Y\} = 1 \\ &\longleftrightarrow X(w) = Y(w) \\ &\quad \text{for all outcomes } w \in \mathcal{S} \text{ such that } \Pr\{w\} \neq 0. \end{aligned}$$

Now Reflexivity and Symmetry are immediate. To prove Transitivity, suppose $\Pr\{X = Y\} = 1 \wedge \Pr\{Y = Z\} = 1$. So, $X(w) = Y(w)$ and $Y(w) = Z(w)$ for all w such that $\Pr\{w\} \neq 0$. By transitivity of equality, $X(w) = Z(w)$ for all w such that $\Pr\{w\} \neq 0$. Hence, $\Pr\{X = Z\} = 1$, as required.

An alternative proof of transitivity that works for general probability spaces—not just the "discrete" spaces considered in 6.042—makes ingenious use of Inclusion-Exclusion:

$$\begin{aligned} \Pr\{X = Z\} &\geq \Pr\{X = Y \wedge Y = Z\} && \text{(monotonicity)} \quad (1) \\ &= \Pr\{X = Y\} + \Pr\{Y = Z\} - \Pr\{X = Y \vee Y = Z\} && \text{(inclusion-exclusion)} \quad (2) \\ &\geq 1 + 1 - 1 = 1. && (3) \end{aligned}$$

So $\Pr\{X = Z\} = 1$. ■