In-Class Problems — Week 10, Wed

Problem 1. Suppose there is a system with n components, and we know from past experience that any particular component will fail in a given year with probability p. That is, letting F_i be the event that the ith component fails within one year, we have

$$\Pr\{F_i\} = p$$

for $1 \leq i \leq n$. The *system* will fail if *any one* of its components fails. What can we say about the probability that the system will fail within one year?

Let F be the event that the system fails within one year. Without any additional assumptions, we can't get an exact answer for $\Pr\{F\}$. However, we can give useful upper and lower bounds, namely,

$$p \leq \Pr\{F\} \leq np. \tag{1}$$

So for example, if $n = 100$ and $p = 10^{-5}$, we conclude that there is at most one chance in 1000 of system failure within a year and at least one chance in 100,000.

Let's model this situation with the sample space $\mathcal{S} ::= \mathcal{P}(\{1, \ldots, n\})$ of subsets of positive integers $\leq n$, where $s \in \mathcal{S}$ corresponds to the numbers of the components which fail within one year. For example, $\{2, 5\}$ is the outcome that the second and fifth components failed within a year and none of the other components failed. So the outcome that the system did not fail corresponds to the emptyset, $\emptyset$.

(a) Show that the probability that the system fails could be as small as p by describing appropriate probabilities for the sample points.

(b) Show that the probability that the system fails could actually could be as large as np by describing appropriate probabilities for the sample points.

(c) Prove the inequality (1).

(WE DIDN'T GET TO THE NEXT TWO PROBLEMS IN CLASS ON WEDNESDAY.)

Problem 2. Smith and Wesson are shooting at a target. Suppose Smith's chance of hitting the target is double that of his friend Wesson, and the probability that at least one of them hits the target is $1/2$. Whether or not one of them hits the target has no effect on the probability that the other one will hit. What is the probability that Wesson hits the target?

Problem 3. Carried over to in-class Problem 1 Friday, Week 10.

Probability Rules

Events

$$
\begin{aligned}
\Pr\left\{\bigcup_{n\in\mathbb{N}} A_n\right\} &= \sum_{n\in\mathbb{N}} \Pr\{A_n\} \text{ for pairwise disjoint } A_n && \text{(Sum Rule)}\\
\Pr\{A - B\} &= \Pr\{A\} - \Pr\{A \cap B\} && \text{(Difference Rule)}\\
\Pr\{\overline{B}\} &= 1 - \Pr\{B\} && \text{(Complement Rule)}\\
\Pr\{A \cup B\} &= \Pr\{A\} + \Pr\{B\} - \Pr\{A \cap B\} && \text{(Inclusion-Exclusion)}\\
\Pr\{A \cup B\} &\le \Pr\{A\} + \Pr\{B\} && \text{(Boole's inequality)}\\
\Pr\{A\} &\le \Pr\{A \cup B\} && \text{(Monotonicity)}
\end{aligned}
$$

[Law of Total Probability] Let $B_0, B_1, \ldots$ be disjoint events whose union is the entire sample space. Then for all events A,

$$\Pr\{A\} = \sum_{i\in\mathbb{N}} \Pr\{A \cap B_i\}.$$

Conditional Probability

$$
\begin{aligned}
\Pr\{A \mid B\} &::= \frac{\Pr\{A \cap B\}}{\Pr\{B\}}\\
\Pr\{A \cap B\} &= \Pr\{A \mid B\}\Pr\{B\} && \text{(Product Rule)}\\
\Pr\{A \mid B\} &= \frac{\Pr\{B \mid A\}\Pr\{A\}}{\Pr\{B\}} && \text{(Bayes Rule)}
\end{aligned}
$$

[Law of Total Probability - Conditional form] Let $B_0, B_1, \ldots$ be disjoint events whose union is the entire sample space. Then for all events A,

$$\Pr\{A\} = \sum_{i\in\mathbb{N}} \Pr\{A \mid B_i\}\Pr\{B_i\}.$$