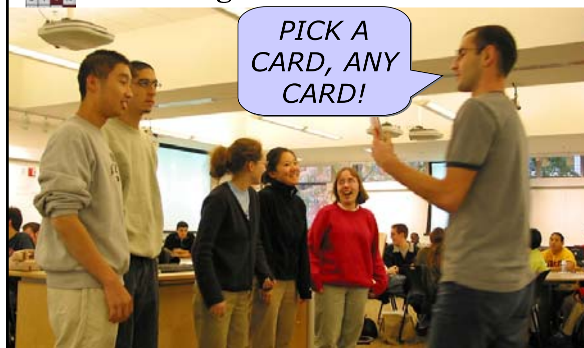
Mathematics for Computer Science

MIT 6.042J/18.062J

Combinatorics II.3

The Magic Trick



The Magic Trick

The Magic Trick

Multisets

Definition. A ***multiset*** is a collection of objects, some of which may not be distinguishable from each other.

Example. $\{a, b, a, c, b, a\} = \{3 \cdot a, 2 \cdot b, 1 \cdot c\}$

repetition counts

Permutations of Multisets

Let S be a multiset with k objects, each with an infinite repetition count. That is,

$$S = \{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_k\} .$$

Q. How many r-permutations of S are there?

A. k^r (easy).

Permutations of Multisets with Limited Repetitions

Q. How many permutations are there of MISSISSIPPI ?

($S = \{ 1{\cdot}M, 4{\cdot}I, 4{\cdot}S, 2{\cdot}P \}$.)

A. **IDEA:** Fill in slots.

Permutations of Multisets with Limited Repetitions

$S = \{ 1{\cdot}M, 4{\cdot}I, 4{\cdot}S, 2{\cdot}P \}$

\# choices for M: $\binom{11}{1}$

× # choices for I: $\binom{10}{4}$

× # choices for S: $\binom{6}{4}$

× # choices for P: $\binom{2}{2}$

= 34,650

S	I	P	M	S	I	I	S	I	S	P

Multinomial Coefficients

Definition. For $n = r_1 + r_2 + \cdots + r_k$, the ***multinomial coefficient*** $C(n; r_1, r_2, \ldots, r_k)$ is defined as

$$\binom{n}{r_1, r_2, \ldots, r_k} = \binom{n}{r_1} \cdot \binom{n - r_1}{r_2} \cdot \binom{n - r_1 - r_2}{r_3} \cdots \binom{n - r_1 - r_2 - \cdots - r_{k-1}}{r_k} = \frac{n!}{r_1!\, r_2! \cdots r_k!}.$$

Permutations of a Multiset with Limited Repetitions

Theorem. The number of permutations of a multiset $\{r_1{\cdot}a_1, r_2{\cdot}a_2, \ldots, r_k{\cdot}a_k\}$, where $n = r_1 + r_2 + \cdots + r_k$ is

$$\binom{n}{r_1, r_2, \ldots, r_k}.$$

Proof. Fill in the n slots, as before. □

In-Class Problems

Problems 1 & 2

Combinations of Multisets with Unlimited Repetitions

Q. How many 2-combinations can be chosen from the multiset
$S = \{\infty{\cdot}a, \infty{\cdot}b, \infty{\cdot}c, \infty{\cdot}d\}$?

A. 10:
{a, a}, {a, b}, {a, c}, {a, d}, {b, b}, {b, c}, {b, d}, {c, c}, {c, d}, {d, d}.

Stars-and-Bars Theorem

Theorem. The number of r-combinations of a multiset with n distinct objects, each with ∞ repetition count, is

$$\binom{n-1+r}{r}.$$

Proof ***(Stars and Bars)***. Define a bijection from the r-combinations of $S = \{\infty\cdot a_1, \infty\cdot a_2, \ldots, \infty\cdot a_n\}$ to the permutations of $T = \{r\cdot★, (n-1)\cdot❙\}$.

Establishing a Bijection

An r-combination of $S = \{\infty\cdot a_1, \infty\cdot a_2, \ldots, \infty\cdot a_n\}$ looks like
$\{r_1\cdot a_1, r_2\cdot a_2, \ldots, r_n\cdot a_n\}$,
where $r = r_1 + r_2 + \cdots + r_n$.

The corresponding permutation of $T = \{r\cdot★, (n-1)\cdot❙\}$ looks like

★★★❙★★❙❙⋯❙★★★ .

$\underbrace{★★★}_{r_1}$ ❙ $\underbrace{★★}_{r_2}$ ❙ $\underbrace{\ }_{r_3}$ ❙ ⋯ ❙ $\underbrace{★★★}_{r_n}$

Double-check: one-to-one, onto.

Example Bijection

How many 2-combinations from

$S = \{\infty\cdot a, \infty\cdot b, \infty\cdot c, \infty\cdot d\}$?

- We have 2 ★'s and 4–1=3 ❙'s:

★★❙❙❙ = {a, a}	❙★❙★❙ = {b, c}
★❙★❙❙ = {a, b}	❙★❙❙★ = {b, d}
★❙❙★❙ = {a, c}	❙❙★★❙ = {c, c}
★❙❙❙★ = {a, d}	❙❙★❙★ = {c, d}
❙★★❙❙ = {b, b}	❙❙❙★★ = {d, d}

Finishing the Proof

Each permutation of $T = \{r\cdot★, (n-1)\cdot❙\}$ can be enumerated by considering $n-1+r$ slots, each filled by a ★ or a ❙, and then choosing r of the slots to hold ❙'s:

★★★❙★★❙❙⋯❙★★★ .

Thus, we have
\# r-combinations of S = # permutations of T

$$= \binom{n-1+r}{r}. \quad \square$$

In-Class Problems

Problem 3