



Combinatorics I.3

Trees, Bijections, & Permutations



Counting Techniques

- Bijections
 - Sum Rule, Inclusion-Exclusion
 - Product Rule
 - Pigeonhole Principle
- Trees
- Permutations



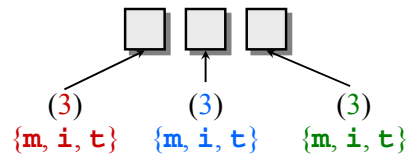
Counting Strings

How many strings of length 3 can be made from the alphabet

$$A = \{m, i, t\} ?$$



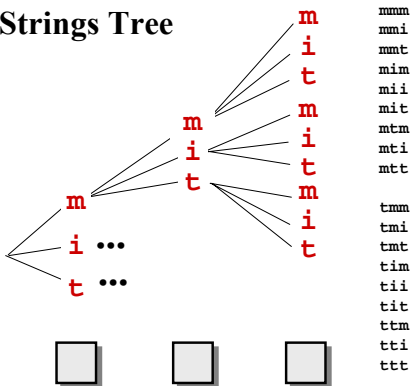
Counting Strings



$$\text{Number of strings} = 3 \cdot 3 \cdot 3 = 27$$



Strings Tree



- mmm
- mmi
- mnt
- mim
- mii
- mit
- mtm
- mti
- mtt
- tmm
- tmi
- tmt
- tim
- tii
- tit
- ttm
- tti
- ttt

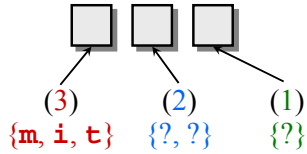


Counting Rearrangements

How many strings of length 3 can be made by *rearranging* the letters **mit**?



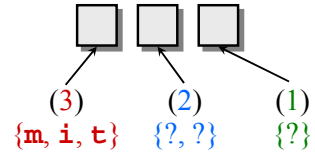
Permutations



Number of strings = $|S| = 3 \cdot 2 \cdot 1 = 6$



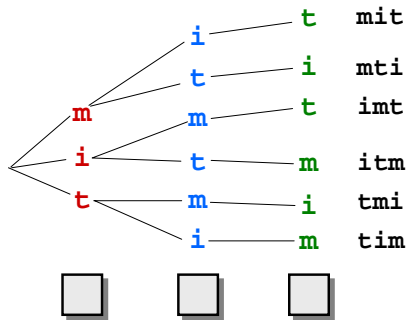
Permutations



$S = \{\text{mit}, \text{mti}, \text{imt}, \text{itm}, \text{tmi}, \text{tim}\}$



Permutations Tree



Comparison

n	n^2	2^n	perms $n!$	strings n^n
1	1	2	1	1
5	25	32	120	3125
10	100	1024	10^7	10^{10}
20	400	10^6	10^{18}	10^{26}
40	1600	10^{12}	10^{47}	10^{64}
100	10^4	10^{30}	10^{157}	10^{200}
1000	10^6	10^{300}	<i>Scheme dies</i>	



In-Class Problems

Problem 1



Important Technique: Matching

Make a **bijection** from the the set we wish to count and a set of known cardinality (or at least one we know how to count...).

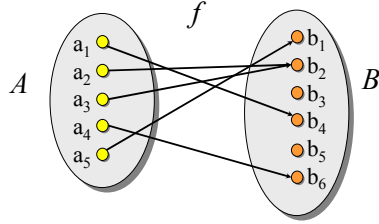
Examples:

- Functions \leftrightarrow Strings
- Lists \leftrightarrow Trees
- Graphs, Relations \leftrightarrow Matrices
- Strings, Polynomials \leftrightarrow Balls-and-Bins

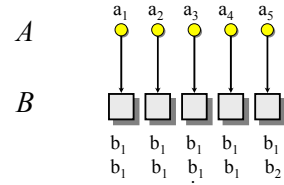


Counting Functions

- How many different functions are there from A to B ?



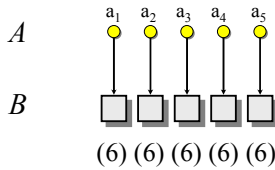
Mapping: Functions as Strings



f All strings of length 5 from the alphabet $\{b_1, b_2, \dots, b_6\}$



Counting Functions

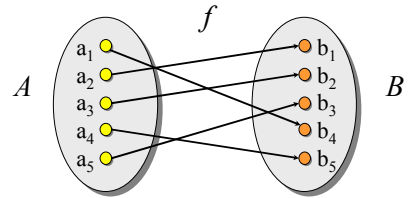


Total number of possible functions
= total number of possible strings
= $6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 = 6^5$

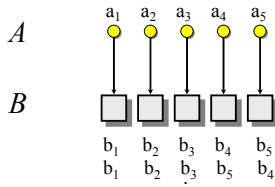


Counting Bijections

- How many different bijections are there from A to B ?



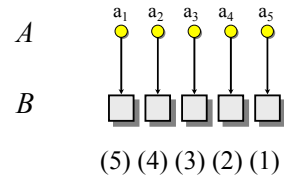
Counting Bijections



f All permutations of 5 letters from the alphabet $\{b_1, b_2, \dots, b_5\}$



Counting Bijections



Total number of possible functions
= total number of possible permutations
= $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$



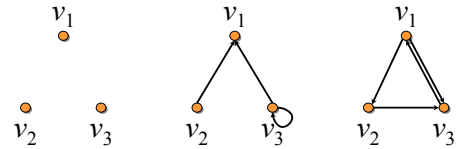
In-Class Problems

Problem 2



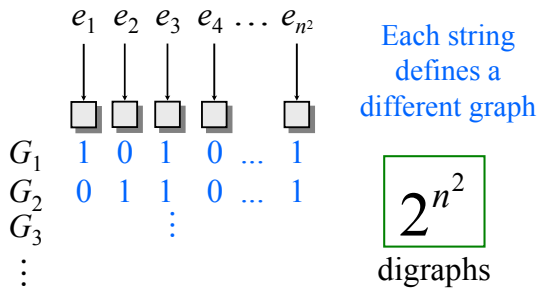
Counting Graphs

How many different directed graphs on n labeled nodes are there?



Graphs as Binary Strings

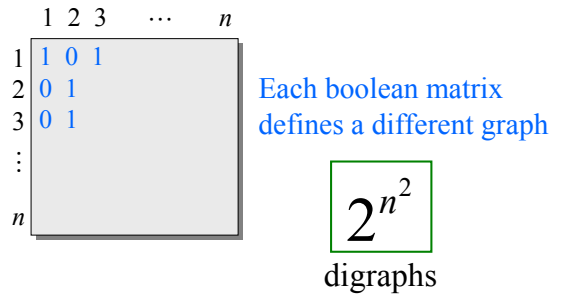
Maximum # edges = n^2



2^{n^2}
digraphs



Graphs as Boolean Matrices



Proving That a Mapping Is a Bijection

To prove that $f: A \rightarrow B$ is **bijective**,

- show that f maps *all* elements of A to *distinct* elements of B
 - i.e.*, f is **one-to-one** or **injective**;
- AND
- show that for *every* element of B , f maps *some* element of A to it
 - i.e.*, f is **onto** or **surjective**.



Proving That Two Sets Have the Same Size

To prove that $|A| = |B|$,

- exhibit an **injective** function $f: A \rightarrow B$ and a **surjective** function $g: A \rightarrow B$, or
- exhibit an **injective** function $f: A \rightarrow B$ and an **injective** function $g: B \rightarrow A$, or
- exhibit a **surjective** function $f: A \rightarrow B$ and a **surjective** function $g: B \rightarrow A$.

6	9	13	7
12	10	5	
3	1	4	11
15	8	14	2

Example: Graph Isomorphism

G_1 and G_2 are *isomorphic* if there is a bijection from the nodes of G_1 to the nodes of G_2 , that *preserves adjacency*.

- *Question*: How many bijections might we need to check to determine whether G_1 and G_2 are isomorphic?
- *Answer*: All possible bijections from the vertices of G_1 to $G_2 = n!$

6	9	13	7
12	10	5	
3	1	4	11
15	8	14	2

In-Class Problems

Problems 3-4