



Combinatorics I.2 Pigeons and Holes



Pigeonhole Principle

If you have *more* pigeons

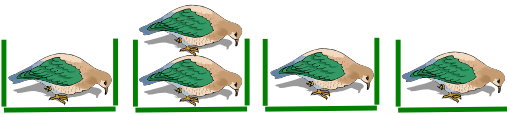


than pigeonholes,



Pigeonhole Principle

Then *some* hole must have *two* pigeons!



Example: 5 Card Draw

Pick a set of 5 cards.

Guaranteed to have

2 with the *same suit*.



5 Card Draw

5 cards (pigeons)



4 suits (holes)



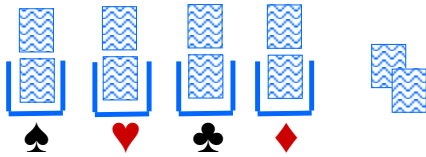
10 Card Draw

How many cards of the *same suit* am I guaranteed to have?

$$\left\lceil \frac{10}{4} \right\rceil = 3$$



10 Card Draw



Cannot have < 3 cards in every hole



Generalized Pigeonhole Principle

If there are n pigeons and t holes, then there will be at least one hole with at least

$$\left\lceil \frac{n}{t} \right\rceil \text{ pigeons}$$



A Pigeonhole Problem

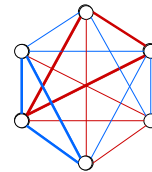
Six people.
Every two are either friends or strangers.

There must be a set of
3 mutual friends or **3 mutual strangers**



Restated as a Graph problem

In a 6-node complete graph where every edge is colored red or blue, there always exists
either a red triangle or a blue triangle



VERIFY THIS!

...by constructing a graph for
your table.

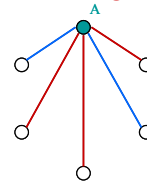


Proof

Vertex A has degree 5

Pigeonhole Principle:

A has at least **3 red edges** or **3 blue edges**

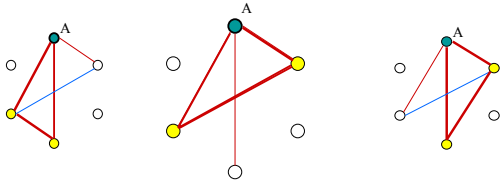


6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Proof

Let A have ≥ 3 red edges.

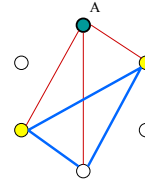
If ends of any two edges are connected by a red edge, then a red triangle is formed



6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Proof

Otherwise, they are connected by blue edges



6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Ramsey Numbers

Theorem: For any natural number k , there exists a n such that a red-blue complete graph on n nodes must have a size k complete blue or red subgraph

$$R(k, k) ::= \text{smallest such } n$$

$$R(3, 3) = 6$$

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

$R(6, 6) = ?$

How many people do you need before you are guaranteed to have a set of six friends or six strangers?

Paul Erdős's answer: If aliens required you answer this in a year, throw in the towel and strike preemptively....

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

In-Class Problems

Problems 1 & 2

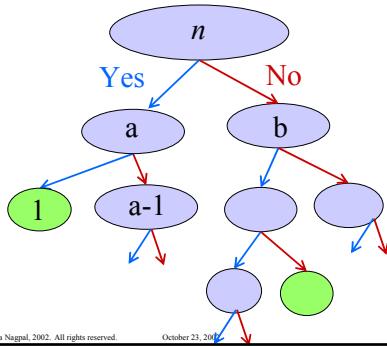
6	9	13	7
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Pick a Number Game

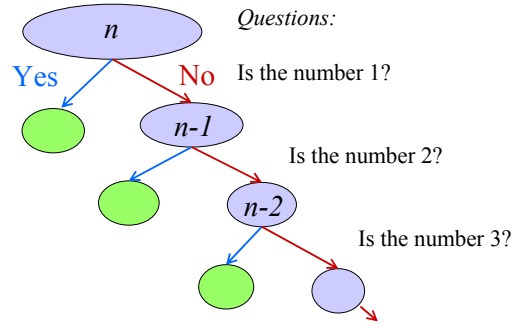
- *Player 1:* Pick a number 1 to Million
- *Player 2:* Can ask Yes/No questions
- How many questions do I need to ask in order to be guaranteed to correctly identify the number?



A Strategy as a Tree



A Possible Strategy



Minimum Number of Questions

- How many questions do I need to ask to guess the number?
 - Worst case, it's the height of the strategy tree
- What is the **minimum number of questions** I could ask and be guaranteed to find out the number?



Pigeonhole Principle

If a tree has $< n$ leaves, then there are **at least 2 numbers** that can not be distinguished.

- The strategy tree must have **at least n leaves**



Shortest Strategy Tree

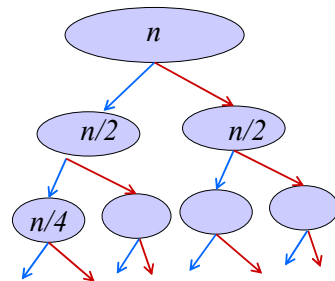
What is the shortest binary tree with n leaves?

- Full Binary Tree
- Height = $\log_2 n$

Lower bound on the number of questions = $\log_2 n$



Optimal Strategy



6	9	13	7
12	10	5	
3	1	4	11
15	8	14	2

Lower Bounds



- How many weighings do you need to find one counterfeit coin amongst 12 coins?
- How many binary comparisons do you need to **sort** n numbers

6	9	13	7
12	10	5	
3	1	4	11
15	8	14	2

Lower Bound on Sorting

- What is the **minimum number** of binary comparisons needed to **sort** n numbers?
- = guessing which permutation of the n numbers you got
- = need $n!$ leaves

6	9	13	7
12	10	5	
3	1	4	11
15	8	14	2

In-Class Problems

Problem 3

6	9	13	7
12	10	5	
3	1	4	11
15	8	14	2

Dilworth's Theorem

If poset with n elements has a longest chain of size t , then there exists an antichain of size at least

$$\left\lceil \frac{n}{t} \right\rceil$$

6	9	13	7
12	10	5	
3	1	4	11
15	8	14	2

Proof

If longest chain has size t , then the n elements can be partitioned into t antichains (*parallel task scheduling thm*)
 So have t antichains (*holes*) and n elements (*pigeons*):

there must exist at least one antichain with

$$\left\lceil \frac{n}{t} \right\rceil \text{ elements (pigeonhole principle)}$$