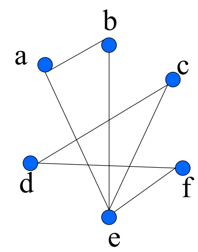
Mathematics for Computer Science

MIT 6.042J/18.062J

Graphs

Model Real World Problems

- Computer Networks
- Airline Connections
- Boston Road Map
- Program Flowchart
- Final Exam Conflicts
- Transistor Layouts
- Game Trees

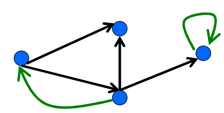


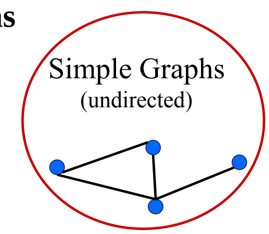
Topics for this week

- Applications of Graphs
- Proving things about Graphs
  – Graph Coloring and Isomorphism
  – Connectivity and Paths
  – Trees, Tours

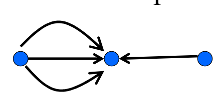
Types of Graphs

Directed Graphs



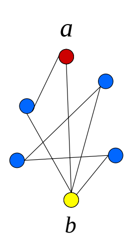


Multi-Graphs



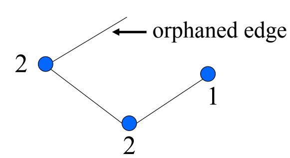
Definition

- **A Simple Graph** is a set of vertices V and a set of edges E where each edge is an unordered pair of distinct vertices a and b.
- **Degree** of a vertex v is the number of edges it connects to.



Exercise: Draw a graph of **3 vertices** with **degrees 2, 2 and 1**

An Impossible Graph



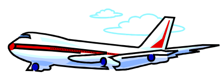
$$\sum_{v \in V} \text{degree}(v) = 2 \times \text{number of edges}$$

- *Theorem:* Sum of degrees must be even
- Handshaking Lemma

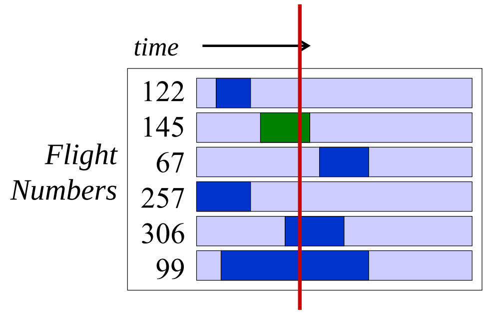
Problem : Airline Gate Allocation

Airline Scheduling:

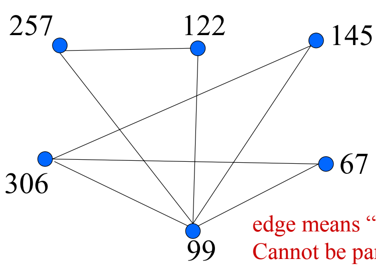
- Given a set of airline flights arriving at different times, how many different gates do I need in order to accommodate them?



Airline Schedule

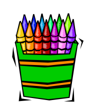


Model as a Graph

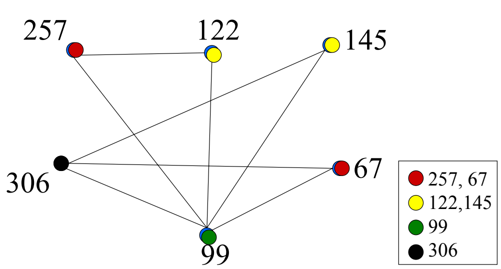


Solution: Graph Coloring

Color each vertex such that no two adjacent vertices have the same color



Assignment of Gates (or Colors)



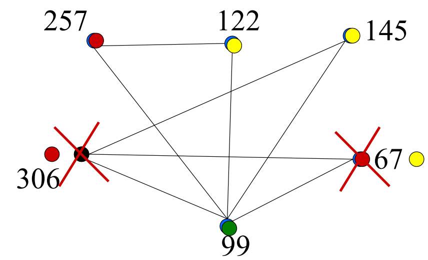
Resource Allocation Problems

- How should I schedule the final exams to avoid two exams being scheduled at the same time if there are students taking both
- How many different habitats do I need to house several species of animals, some of which cannot coexists with others
- Frequency allocation: How many different frequencies do I need for overlapping radio broadcasts of wireless devices?

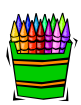
Chromatic Number

- *Question:* what is the minimum number of colors I need?
  - *Chromatic Number*
- How do I know that this is the minimum?

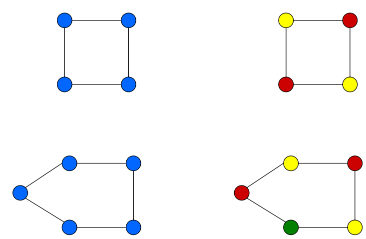
Assignment Using Fewer Gates



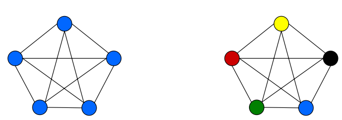
Class Problem 1



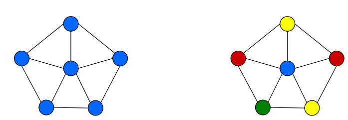
Simple Cycles C_n



Complete Graph K_n



Wheel W_n

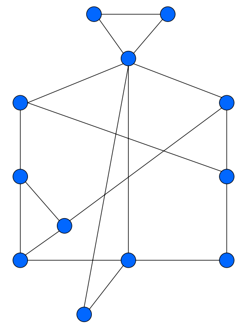


Chromatic Number

- Odd cycles need 3 colors
- Complete graph needs exactly *n* colors
- Wheels can be colored with 4 colors
  - (if outer rim is even, then 3 colors)

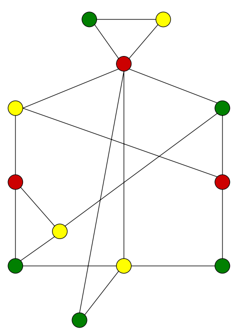
- What about arbitrary graphs ?

 L4-1.19



Courtesy
Steven Rudich
CMU

 L4-1.20



- Prove that it *can* be colored with 3 colors

- Prove that it *can not* be colored with 2 colors

 L4-1.21

Arbitrary Graphs

Inspiration vs Perspiration

2-colorable?
Odd cycle exists: NO
No odd cycle: YES!

3-colorable?
Brute-force approach, test all possible assignments
3^n
Can't do much better!

Minimum color?
– Exhausting idea…

 L4-1.22

Brute Force vs Inspiration

```
Scheme saved on Sunday November 21, 1993 at 9:15:23 PM
  Release 7.3.1
  Microcode 11.146
  Runtime 14.166

1 ]=> (expt 3 2)
;Value: 9

1 ]=> (expt 3 10)
;Value: 59049

1 ]=> (expt 3 20)
;Value: 3486784401

1 ]=> (expt 3 100)
;Value: 515377520732011331036461129765621272702107522001
```

 L4-1.23

Graph Coloring Theorems

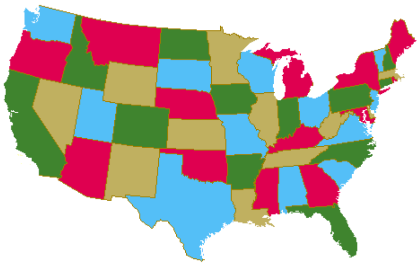
- A graph is 2-colorable iff there are no odd cycles.
- Complete graph K_n requires *n* colors

- If the maximum degree is d_{max}, then the graph can be colored with $(d_{max} + 1)$ colors

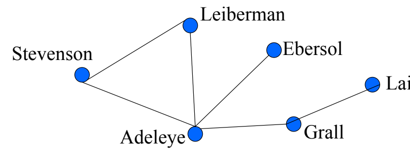
- All planar graphs can be colored with 4 colors

 L4-1.24

Four Color Theorem



Who Knows Who Graph



Six Degrees of Separation

– Is there a path of length ≤ 6 between any two nodes?

Oracle of Bacon
(a.k.a. Six Degrees of Bacon)

(University of Virginia)

Total number of linkable actors: 535914

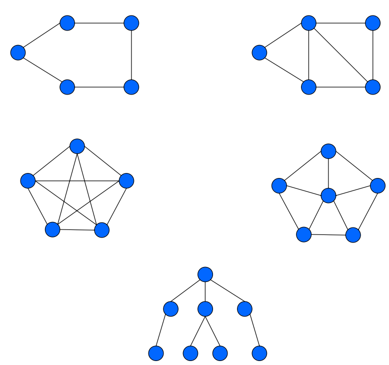
Bacon Number	# of People
0	1
1	1620
2	124832
3	326664
4	76190
5	5825
6	652
7	104
8	24
9	1
10	1

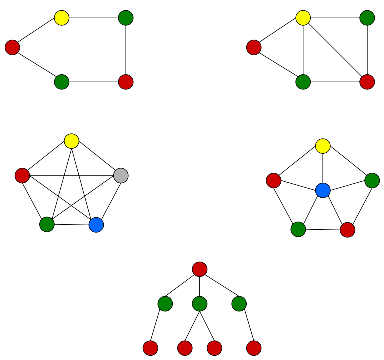
Class Problems

Coloring with d_{max} colors

- Induction Hypothesis

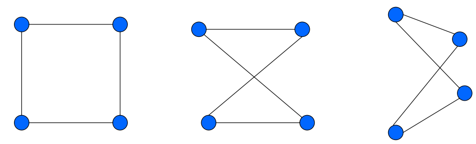
  P(n) = a graph with *n* vertices and maximum degree d_{max} can be colored with $d_{max} + 1$ colors
- Base Case
- Inductive Step
  - Given an $n+1$ graph, remove one vertex
  - Remaining n graph is colorable in $d_{max} + 1$ colors
  - (why? Need to prove that the n graph has d_{max} degree)
  - Add vertex back in, must be one left over color
  - (why?)





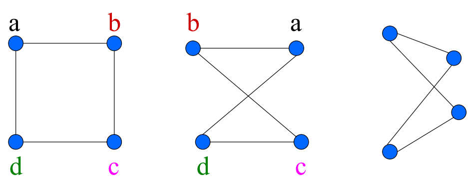
Graph Isomorphism

- Topology, not Geometry



Graph Isomorphism

- Bijection between G1 and G2 nodes that preserves the adjacency list



Graph Isomorphism

- Isomorphism is an equivalence relation on the set of *n*-vertex Graphs!
- Can test for some invariants but otherwise another hard problem — *n!* possible mappings

Representations

List $((v_1, v_1)\,(v_1, v_3)\,(v_3, v_3))$

Matrix

	v_1	v_2	v_3
v_1	1	0	1
v_2	0	0	0
v_3	0	0	1

Adjacency Lists

$v_1 \rightarrow v_1, v_3$

$v_3 \rightarrow v_3$