

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Relations and Graphs

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Partial Order



Two students are related to each other if one is **shorter** and **younger** than the other

$$(s_1, a_1) R (s_2, a_2) \text{ iff } (s_1 \leq s_2) \wedge (a_1 \leq a_2)$$

- Reflexive
- Antisymmetric
- Transitive

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Partial Order



Transitive:

Prove $(s_1, a_1) R (s_2, a_2)$ and $(s_2, a_2) R (s_3, a_3)$ implies $(s_1, a_1) R (s_3, a_3)$:

$$(s_1 \leq s_2 \leq s_3) \text{ and } (a_1 \leq a_2 \leq a_3)$$

Therefore $(s_1 \leq s_3) \text{ and } (a_1 \leq a_3)$

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Chain

If we line up the students in a chain in order of *increasing height*, then they are also *increasing in age*.

$$(s_1, a_1), (s_2, a_2), \dots, (s_n, a_n)$$

Consecutive pairs (s_k, a_k) and (s_{k+1}, a_{k+1}) are related by R , so

$$\text{when } (s_k \leq s_{k+1}), \text{ then also } (a_k \leq a_{k+1})$$

6	9	13	7
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15	8	11	2

AntiChain

If we line up the students in an antichain in order of *increasing height*, then they are *decreasing in age*.

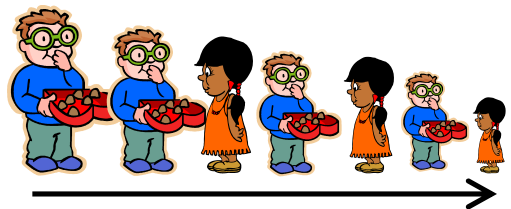
$$(s_1, a_1), (s_2, a_2), \dots, (s_n, a_n)$$

Any pairs (s_i, a_i) and (s_k, a_k) are **not** R -related, so

$$\text{when } (s_i \leq s_k), \text{ then } (a_i > a_k)$$

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

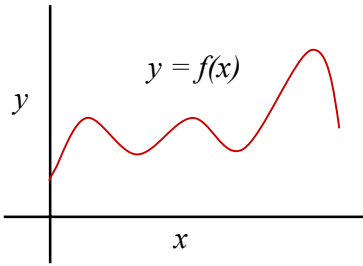
Dilworth Demo



Older

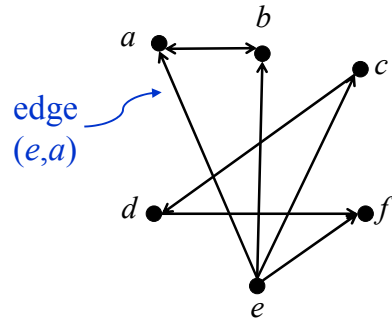
6	9	13	7
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3	1	4	14
15	5	11	2

Normal Person's Graph



6	9	13	7
12	10	8	
3	1	4	14
15	5	11	2

Computer Scientist's Graph



6	9	13	7
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3	1	4	14
15	5	11	2

Definition

A **Graph** is a set of vertices V and a set E of edges, such that

$$E \subseteq V \times V$$

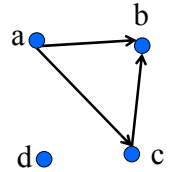
So E is simply a **binary relation** on the set V .

6	9	13	7
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Relations and Graphs

$$A = \{a, b, c, d\}$$

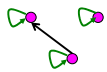
$$R = \{(a, b) (a, c) (c, b)\}$$



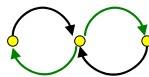
6	9	13	7
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3	1	4	14
15	5	11	2

Properties of Relations

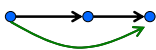
Reflexive



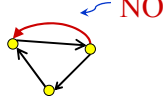
Symmetric



Transitive



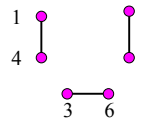
Antisymmetric



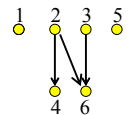
6	9	13	7
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3	1	4	14
15	5	11	2

Types of Relations

Equivalence (mod 3)



Partial Order
($a \mid b$)



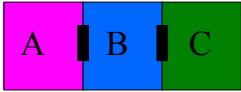
Total Order ($<$)



6	9	13	7
12	10	8	
3	1	4	14
15	11	5	2

A Relation on Buildings

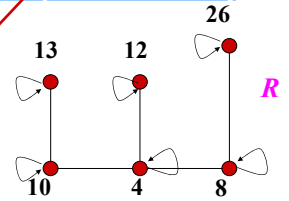
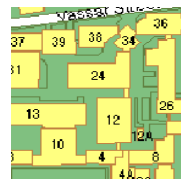
$a R b ::=$ Building a is “physically adjacent” to Building b



$A R B, B R A$
not $A R C$
 $A R A$

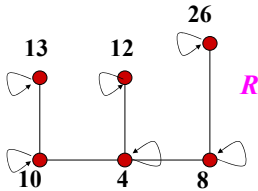
6	9	13	7
12	10	8	
3	1	4	14
15	11	5	2

MIT Building Connections



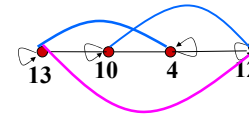
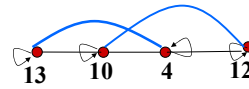
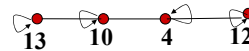
6	9	13	7
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3	1	4	14
15	11	5	2

Class Problem 1



6	9	13	7
12	10	8	
3	1	4	14
15	11	5	2

MIT Building Connections



$R^2 = \{(a,b) \mid a,b \text{ are connected by path of length } 2\}$

$R^3 = \{(a,b) \mid a,b \text{ are connected by a path of length } 3\}$

6	9	13	7
12	10	8	
3	1	4	14
15	11	5	2

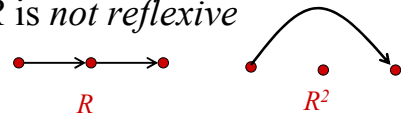
Composition and Path Lengths

R^k is the set of all pairs (a,b) such that a and b are **connected** by a **path of length exactly k** .

6	9	13	7
12	10	8	
3	1	4	14
15	11	5	2

Composition and Path Lengths

If R is *not reflexive*



If R is *reflexive and transitive*



6	9	13	7
12	10	8	
3	1	4	14
15	5	11	2

The Same Questions

Question 1: Can you drive from one state to another with at most 5 state-boundary crossings?

$$C \times C = R^0 \cup R^1 \cup \dots \cup R^5?$$

Question 2: Can you fly on KLM from Boston to Paramaribo with at most 3 stopovers?

$$(BOS, PAR) \in R^0 \cup R^1 \cup \dots \cup R^4?$$

Quiz: *Paramaribo is the capital of ...?*

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Connectivity

Is it possible at all to get from bldg a to b ?

Is there a path of some length k from a to b ?

$$(a, b) \in \bigcup_{k=0}^{\infty} R^k?$$

$$= R^{n-1}$$

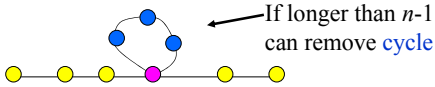
Why $n-1$?

R is reflexive and ...

6	9	13	7
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Connectivity

...the greatest distance between any pair of nodes is $n-1$:



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Reflexive Transitive Closure

$$R^* ::= \bigcup_{k=0}^{\infty} R^k$$

$$= \underbrace{R^0}_{Id} \cup R^1 \cup R^2 \cup \dots \cup R^k \cup \dots$$

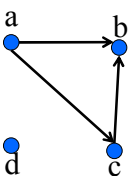
aka the **Connectivity Relation**

6	9	13	7
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3	1	4	14
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Boolean Matrix Representation

$$A = \{a, b, c, d\}$$

$$R = \{(a, b) (a, c) (c, b)\}$$



	a	b	c	d
a	0	1	1	0
b	0	0	0	0
c	0	1	0	0
d	0	0	0	0

6	9	13	7
12	10	8	
3	1	4	14
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Boolean Matrix Operations

$$\text{e.g } \bar{R} = A \times A - R$$

(all pairs not in R)

	a	b	c	d
a	0	1	1	0
b	0	0	0	0
c	0	1	0	0
d	0	0	0	0

 \Rightarrow

	a	b	c	d
a	1	0	0	1
b	1	1	1	1
c	1	0	1	1
d	1	1	1	1

6	9	13	7
12	10	5	
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Composition using Matrices

$$\begin{matrix} & \begin{matrix} b_1 & b_2 & b_3 & b_4 \end{matrix} \\ \begin{matrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix} \circ \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix} = \begin{matrix} & \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{matrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

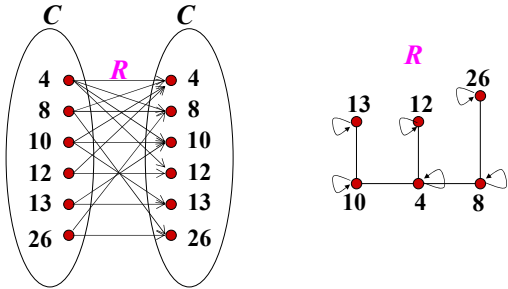
$$T(a_1, c_1) = [R(a_1, b_1) \wedge S(b_1, c_1)] \vee [R(a_1, b_2) \wedge S(b_2, c_1)] \vee [R(a_1, b_3) \wedge S(b_3, c_1)] \vee [R(a_1, b_4) \wedge S(b_4, c_1)]$$

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Class Problems

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

MIT Building Connections



6	9	13	7
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MIT Building Connections

